

Full Lagrangian and Hamiltonian for quantum strings on $\text{AdS}_4 \times \mathbb{CP}^3$ in a near plane wave limit

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Abstract

We find the full interacting Lagrangian and Hamiltonian for quantum strings in a near plane wave limit of $\text{AdS}_4 \times \mathbb{CP}^3$. The leading curvature corrections give rise to cubic and quartic terms in the Lagrangian and Hamiltonian that we compute in full. The Lagrangian is found as the type IIA Green-Schwarz superstring in the light-cone gauge employing a superspace construction with 32 grassmann-odd coordinates. The light-cone gauge for the fermions is non-trivial since it should commute with the supersymmetry condition. We provide a prescription to properly fix the κ -symmetry gauge condition to make it consistent with light-cone gauge. We use fermionic field redefinitions to find a simpler Lagrangian. To construct the Hamiltonian a Dirac procedure is needed in order to properly keep into account the fermionic second class constraints. We combine the field redefinition with a shift of the fermionic phase space variables that reduces Dirac brackets to Poisson brackets. This results in a completely well-defined and explicit expression for the full interacting Hamiltonian up to and including terms quartic in the number of fields.

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1 Introduction and summary

For the last decade, the duality between four-dimensional $\mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory and type IIB string theory on $\text{AdS}_5 \times S^5$ has been celebrated as the one example of an exact duality between gauge theory and string theory [1]. Only last year Aharony, Bergman, Jafferis and Maldacena (ABJM), inspired by earlier work on superconformal Chern-Simons theories [2], proposed a new exact duality between a Chern-Simons-matter gauge theory and M-theory compactified on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ [3]. In a particular limit the gauge theory is dual to type IIA string theory compactified on $\text{AdS}_4 \times \mathbb{CP}^3$. In this region in the parameter space, the new duality is between three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory (ABJM theory) and type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ which preserves 24 out of the 32 supersymmetries. ABJM theory has $U(N) \times U(N)$ gauge symmetry with

Chern-Simons like kinetic terms at level k and $-k$ and it is weakly coupled when the 't Hooft coupling $\lambda = N/k$ is small. Instead type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ is a good description when $1 \ll \lambda \ll k^4$.

As for the $\text{AdS}_5/\text{CFT}_4$ duality, there is some evidence for integrability in the planar limit also for the $\text{AdS}_4/\text{CFT}_3$ duality [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In particular an all-loop asymptotic Bethe ansatz has been proposed [9, 16]. Recently a set of functional equations in the form of a Y-system based on the integrability of the superstring σ -model, which defines the anomalous dimensions of local single trace operators, has been formulated also for the $\text{AdS}_4/\text{CFT}_3$ duality [17].

One of the most important results in the study of integrability of the $\text{AdS}_5/\text{CFT}_4$ correspondence was the calculation of the complete set of first-order curvature corrections to the spectrum of light-cone gauge string theory that arises in the expansion of $\text{AdS}_5 \times S^5$ about the plane-wave [18, 19], the so-called “near BMN limit” [20]. Among other results, this has produced the first evidence of the famous “three loop discrepancy”, which was then understood and solved by the inclusion of the dressing factor that interpolates between weak and strong coupling, in the Bethe equations that describe the spectrum of the gauge and the string theory [21].

Analogous calculations for the $\text{AdS}_4/\text{CFT}_3$ duality were initiated in [10] (see also [12]) where the spectrum of two bosonic oscillator states in the $SU(2) \times SU(2)$ sector was computed and compared with the solutions of the proposed all loop Bethe equations [9, 10]. In order to perform a complete analysis of the spectrum of string oscillators around a pp-wave limit of $\text{AdS}_4 \times \mathbb{CP}^3$ it is however necessary to include all the bosonic and fermionic directions in the computation of the interacting Lagrangian and Hamiltonian.

In this paper, using the $\text{AdS}_4 \times \mathbb{CP}^3$ superspace construction for type IIA superstrings of refs. [22, 23], we provide the full interacting Lagrangian and Hamiltonian for the type IIA Green-Schwarz superstring in the light cone gauge in a near plane wave limit of $\text{AdS}_4 \times \mathbb{CP}^3$. The near plane wave background is given by the leading pp-wave background plus the $1/R$ and $1/R^2$ curvature corrections, R being the radius of \mathbb{CP}^3 . We find all the terms in the Lagrangian and Hamiltonian which are quadratic, cubic and quartic in the number of fields. The Penrose limit defining the near plane wave background follows a null geodesic that moves along one of the isometries of \mathbb{CP}^3 [8]. We can thus use our Hamiltonian to find the leading finite-size corrections to the spectrum of a string fluctuating around the null geodesic.

In the quantization of type IIA string theory on the $\text{AdS}_4 \times \mathbb{CP}^3$ background there are many non trivial issues that should be carefully addressed and solved in order to have a complete characterization of the spectrum.

- (a) The κ -symmetry gauge choice for the light-cone gauge on this background cannot be chosen as in $\text{AdS}_5 \times S^5$ or in flat space. The less than maximal supersymmetry means that the light-cone quantization for the fermions will be completely different than that for the $\text{AdS}_5 \times S^5$ background since it should commute with the supersymmetry condition.
- (b) Fermionic field redefinitions are conveniently used to find a simpler Lagrangian such that the fermionic momenta have no $1/R^2$ corrections. This simplifies the phase space

variables and makes it easier to construct the Hamiltonian.

- (c) A Dirac procedure is needed in order to properly keep into account the fermionic second class constraints. This is highly non-trivial in this case since there are both first and second order curvature corrections to the Hamiltonian that should be taken into account. The Dirac brackets can then be reduced to Poisson brackets by a suitable field redefinition.
- (d) We prove that the two-fermion terms in the Lagrangian that arises from the superspace model, match those found using the general type IIA Green-Schwarz Lagrangian of [24].
- (e) There is strong evidence that the terms in the Hamiltonian which are quartic in the fields are not normal ordered. This is because there are divergences coming from the cubic terms in the Hamiltonian when used at the second order in perturbation theory for certain string states [10, 25] which can only be canceled by the inclusion of the appropriate normal ordering functions.

Let us comment on these points. With respect to (a), the κ -symmetry gauge condition $\Gamma^+\theta = 0$ where θ is a 32-component spinor and $\Gamma^+ = \Gamma^0 + \Gamma^9$, which is used for flat space and $\text{AdS}_5 \times S^5$, cannot be used indiscriminately on $\text{AdS}_4 \times \mathbb{CP}^3$. The isometry group of \mathbb{CP}^3 is $SU(4)$, it preserves not only the \mathbb{CP}^3 metric, g_{ab} , but also the Kähler form, J_{ab} . Thus the vacuum is $SU(4)$ invariant. This symmetry of the ten dimensional theory is dual to the R -symmetry of the $\mathcal{N} = 6$ supersymmetry of the gauge theory. To see how supersymmetry is realized one can study supersymmetry transformation laws [26]. In the purely bosonic ground state of the 10-dimensional theory given by $\text{AdS}_4 \times \mathbb{CP}^3$ the fermion fields are set to zero, thus the criterion for unbroken supersymmetry of the vacuum is that their supersymmetric variations should also vanish, $\delta\theta = 0$. In order to see when this condition is realized, one can define a quantity out of the $SU(4)$ invariant Kähler form, which takes the following explicit form in terms of 32 dimensional gamma matrices

$$J = \Gamma_{0123}\Gamma_{11}(-\Gamma_{49} - \Gamma_{56} + \Gamma_{78}) = \Gamma_{5678} - \Gamma_{49}(\Gamma_{56} - \Gamma_{78}) \quad (1.1)$$

where we choose for \mathbb{CP}^3 the directions 4 to 9. One can easily show that $J^2 = 2J + 3$ and hence that J has 24 eigenvalues -1 and 8 eigenvalues 3. It was shown already in [26] that in the vacuum defined by $\text{AdS}_4 \times \mathbb{CP}^3$, the condition $\delta\theta = 0$ is realized only by the 24 eigenvalues $J = -1$. This is the reason why $\text{AdS}_4 \times \mathbb{CP}^3$ preserves only 24-supersymmetries. The projector on to supersymmetric states is then simply $P = (3 - J)/4$. Now, the light-cone gauge condition $\Gamma^+\theta = 0$ does not commute with J , and thus with the projector P , and there is no choice of path along \mathbb{CP}^3 for which this could happen since the Γ^a with $a = 4, 5, 6, 7, 8, 9$ do not commute with P . Thus on $\text{AdS}_4 \times \mathbb{CP}^3$ the standard light-cone gauge condition is not always consistent with supersymmetry.

We shall show in this paper how to properly fix the κ -symmetry gauge condition on this background in such a way that it is consistent with our choice of light-cone gauge. This will be done for string states in a Penrose limit defining the near plane wave background that follows a null geodesic that moves along one of the isometries of \mathbb{CP}^3 [8]. In particular

even to derive the pp-wave spectrum one should in principle use the appropriate light-cone gauge condition. All the papers in the literature that derived the pp-wave spectrum on this background [27, 28, 29, 5] used the standard $\Gamma^+\theta = 0$ condition that, even if provides the correct spectrum for the pp-wave, is, in principle, inconsistent with κ -symmetry. Whereas for the pp-wave spectrum this has proven not to be a real issue, a correct, supersymmetry preserving, gauge fixing becomes crucial in deriving the curvature corrections to the spectrum.

As stated in point (b) it is convenient to first perform field redefinitions on the Lagrangian such that the fermionic momenta have no $1/R^2$ corrections. This has the advantage that one needs only to take into account the $1/R$ corrections to the fermionic momenta when changing variables to fermionic phase space variables in the Hamiltonian and when performing the Dirac procedure.

With respect to (c), the set of constraints that arise from the definitions of the fermionic momenta for the Green-Schwarz type IIA superstring on the $\text{AdS}_4 \times \mathbb{CP}^3$ background are second-class. This means that to make a consistent quantization the quantum anticommutator of two fermionic fields should be identified with their Dirac bracket (which depends on the Poisson bracket algebra of the constraints) rather than with their Poisson bracket. For type IIA superstrings on $\text{AdS}_4 \times \mathbb{CP}^3$, at variance with what happens for type IIB superstrings on the $\text{AdS}_5 \times S^5$ background, the canonical commutation relations have a complicated structure due to the fact that the Dirac brackets receive both $1/R$ and $1/R^2$ corrections. This makes canonical quantization of the Hamiltonian a much harder problem. Fortunately, one can circumvent this by making a field redefinition of the fermionic phase space variables which thus changes the Hamiltonian. We find in this paper a particularly elegant way to make this field redefinition, which is both first- and second-order in the curvature correction, by combining it with the initial field redefinition that one should perform to write out the Hamiltonian in fermionic phase space variables in going from the Lagrangian to the Hamiltonian. Moreover, it is particular simple since we write the combined field redefinition in terms of 32-dimensional spinors. We find thus an elegant way to resolve these problems such that we can compute the final complete Hamiltonian. The cubic and quartic fermionic Hamiltonian, the main results of this paper, are given in Eqs.(7.50,7.51).

With respect to (d), our results are that we find expressions for the Green-Schwarz superstring Lagrangian and Hamiltonian in the full 32 dimensional spinor space and that we derive a consistent light-cone gauge fixing for states belonging to \mathbb{CP}^3 . The quadratic part of the fermionic action of [24] is sufficient to make one-loop computations around configurations as folded spinning string in $\text{AdS}_4 \times \mathbb{CP}^3$ [30], but for a complete quantum calculation of the finite-size corrections to the spectrum of strings states, the full Hamiltonian is needed including all terms that are quadratic, cubic and quartic in the number of fields.

With respect to (e), our paper obviously builds on early papers [10, 12] where the bosonic string spectrum was examined in the $SU(2) \times SU(2)$ sector using only the bosonic Hamiltonian by employing zeta-function regularization to regularize divergences coming from the cubic Hamiltonian at the second order in the perturbative expansion in the inverse of the curvature radius R . In those papers it was assumed that the four-field terms in the Hamiltonian are normal ordered. However, this seems on further scrutiny not a valid assumption since

the divergent contributions coming from second order perturbation theory are, on general grounds, always negative and cannot be canceled by analogous terms coming from the cubic Hamiltonian with two fermion and one boson fields [25]. Thus, one should include all the non-normal-ordered terms also in the four bosons, four fermions and two-fermion-two-boson parts of the full Hamiltonian to obtain the correct spectrum ¹.

The explicit construction of the complete $\text{AdS}_4 \times \mathbb{CP}^3$ sigma model including all the 32 Grassmann-odd coordinates was done in refs. [22, 23]. Whereas in the maximally supersymmetric $\text{AdS}_5 \times S^5$ background the supergeometry is described by the coset superspace $SU(2, 2|4)/SO(5) \times SO(1, 4)$ one instead has that the type IIA $\text{AdS}_4 \times \mathbb{CP}^3$ superspace is not a coset superspace. Its supergeometry can be completely characterized by the $OSp(6|4)/U(3) \times SO(1, 3)$ coset superspace only on a submanifold of the superspace [6, 7]. On this submanifold the classical superstring equations of motion are integrable [6, 7], generalizing the corresponding result for type IIB superstring propagating on the $\text{AdS}_5 \times S^5$ supercoset [32]. We are using the type IIA Green-Schwarz action of refs. [22, 23] on this particular submanifold. However, in the $\text{AdS}_4 \times \mathbb{CP}^3$ superspace there is a different submanifold described by a “twisted” $OSp(2|4)/SO(2) \times SO(1, 3)$ superspace, which is not a supercoset, and the ingredients used to prove integrability in [32] do not directly apply to this sector of the theory. Therefore, it remains an open problem to determine whether the complete set of classical equations of motion of the Green-Schwarz superstring propagating on the $\text{AdS}_4 \times \mathbb{CP}^3$ superspace is even classically integrable. The fact that the $\text{AdS}_4 \times \mathbb{CP}^3$ superspace with 32 fermionic directions is not a supercoset requires in fact more general techniques to prove classical integrability.

Several papers have developed the Lagrangian (and in some cases the Hamiltonian) for the superspace construction by using a 24 dimensional spinor space that is manifestly supersymmetric [6, 7, 33, 34, 35, 31]. In particular in [31] the four-fermion Hamiltonian is found, however, the complete quartic Hamiltonian has not been computed, only preliminary versions of the interacting Lagrangian and Hamiltonian have been provided. In order to have a complete characterization of the spectrum, it is necessary to derive these objects carefully dealing with all the issues we described above. This is what we do in this paper.

Our main motivations for this work come from some interesting differences between the $\text{AdS}_4/\text{CFT}_3$ duality and the $\text{AdS}_5/\text{CFT}_4$ duality:

- (1) The magnon dispersion relation in the $\text{AdS}_4/\text{CFT}_3$ duality can vary as a function of λ . Indeed, shortly after the discovery of the $\text{AdS}_4/\text{CFT}_3$ duality it was found that a magnon in the $SU(2) \times SU(2)$ sector of ABJM theory has a dispersion relation that depends non-trivially on the coupling [5, 8, 29]

$$\Delta = \sqrt{\frac{1}{4} + h(\lambda) \sin^2\left(\frac{p}{2}\right)}, \quad h(\lambda) = \begin{cases} 4\lambda^2 + \mathcal{O}(\lambda^4) & \text{for } \lambda \ll 1 \\ 2\lambda + \mathcal{O}(\sqrt{\lambda}) & \text{for } \lambda \gg 1 \end{cases} \quad (1.2)$$

where the weak coupling result is from [4, 5]. Corrections to the leading weak and strong coupling results have been discussed in [13, 14, 36, 37, 38, 39, 30, 40].

¹These crucial terms were ignored in the analysis of ref. [31]

- (2) In the $\text{AdS}_4/\text{CFT}_3$ duality one has $4_B + 4_F$ magnons, *i.e.* four bosonic and four fermionic magnons, in the Bethe ansatz for ABJM theory. However, the pp-wave background has $8_B + 8_F$ magnons. $4_B + 4_F$ of these (the light magnons) correspond to the $4_B + 4_F$ magnons in the Bethe ansatz. The other $4_B + 4_F$ magnons (the heavy magnons) should instead somehow emerge from the spectrum of the light magnons. This is discussed in [41, 31].
- (3) While the $\mathcal{N} = 4$ SYM theory and the $\text{AdS}_5 \times S^5$ background have the maximally possible amount of supersymmetries with 32 supercharges preserved, the $\mathcal{N} = 6$ superconformal Chern-Simons theory of ABJM and the $\text{AdS}_4 \times \mathbb{CP}^3$ background have 24 supercharges preserved. This allows for the radius of $\text{AdS}_4 \times \mathbb{CP}^3$ to vary as a function of λ [42].

With respect to both (1) and (2) it is very important to find the leading finite-size corrections to the quantum string spectrum. Finding the quantum string spectrum including in particular also the non-normal-ordered terms in the Hamiltonian should yield a finite spectrum for the quantum string without need of regularizing. With respect to (1) this will settle the issue of what the $1/\sqrt{\lambda}$ correction to $h(\lambda)$ is for large λ . This is an important question since it has been found that there are certain semi-classical spinning string configurations for which the one-loop correction to the leading energy can only match with the all-loop Bethe ansatz of [37, 38, 39, 30, 40] provided there is a certain non-zero value for this $1/\sqrt{\lambda}$ correction. Instead other calculations [43, 44] have found that this correction should be zero. It is even speculated if this number is measurable, or if it is scheme dependent, since one can make redefinitions of the coupling. However, if the quantum string spectrum reveals the same answer for this correction as the one-loop correction to the semi-classical string configuration it would suggest that string theory picks out a unique value. For the spinning string this matter has been thoroughly discussed in [30]. We postpone the computation of the quantum string spectrum to a later publication [25].

With respect to (2), it would be important to examine how the heavy $4_B + 4_F$ magnons in the pp-wave background can emerge from the $4_B + 4_F$ light magnons in the Bethe ansatz. A proposal for how this works in the continuum limit is presented in [41]. However, it is not immediately clear how this proposal should resolve the problem for the discrete spectrum of the quantum string. Certainly, the completely well-defined and explicit expressions for the full interacting Hamiltonian that we provide in this paper, allowing for a complete calculation of the oscillator spectrum, will shed some light on this subtle problem.

This paper is built up as follows. In Section 2 we compute the Lagrangian for the type IIA Green-Schwarz (GS) superstring in $\text{AdS}_4 \times \mathbb{CP}^3$ using the superspace construction of [22, 23]. In Section 3 we prove the equivalence to the general type IIA two-fermion Lagrangian of ref. [24]. In Section 4, we analyze the light-cone gauge and the corresponding fixing of κ -symmetry. In Section 5 we find the pp-wave Lagrangian and Hamiltonian and derive the pp-wave spectrum for the light and heavy modes. In Section 6 we provide the field redefinitions on the Lagrangian necessary to pass to the Hamiltonian formalism. Finally Section 7 contains our results for the full light-cone Hamiltonian up to terms quartic in the number of fields.

Appendix A contains the details of the $\text{AdS}_4 \times \mathbb{CP}^3$ background along with the near plane wave limit. Our conventions for the Gamma-matrices are instead given in Appendix B. In Appendix C we derive the structure constants of the $OSp(6|2,2)$ algebra and the fermionic matrix entering in the four fermion terms of the Lagrangian.

2 Lagrangian for type IIA superstring from superspace

In this section we present the Lagrangian for the type IIA Green-Schwarz (GS) superstring in $\text{AdS}_4 \times \mathbb{CP}^3$ using the superspace construction of [22, 23]. We restrict ourselves to the supersymmetric fermionic directions. We consider the light-cone gauge and the corresponding fixing of κ -symmetry in Section 4. The $\text{AdS}_4 \times \mathbb{CP}^3$ background is presented in Appendix A. The Gamma-matrix conventions are presented in Appendix B.

2.1 Supersymmetric fermionic directions

For the type IIA GS string we have two Majorana-Weyl spinors $\theta^{1,2}$ with opposite chirality, *i.e.* $\Gamma_{11}\theta^1 = \theta^1$ and $\Gamma_{11}\theta^2 = -\theta^2$. We collect these into a 32 component real spinor $\theta = \theta^1 + \theta^2$.

In the superspace construction of [22, 23] all the 32 real fermionic directions of θ are considered. 24 of these are supersymmetric and 8 are non-supersymmetric. We shall restrict ourselves to the 24 directions which are supersymmetric since we are interested in considering curvature corrections the pp-wave background that comes from a Penrose limit corresponding to a null geodesic moving on an isometry of \mathbb{CP}^3 . This means that we can choose a gauge for the κ -symmetry of the type IIA GS Lagrangian where the 8 non-supersymmetric fermionic directions are put to zero [6, 22, 23].

The 24 supersymmetric directions are given as follows. Recall the matrix J of Eq. (1.1)

$$J = \Gamma_{0123}\Gamma_{11}(-\Gamma_{49} - \Gamma_{56} + \Gamma_{78}) = \Gamma_{5678} - \Gamma_{49}(\Gamma_{56} - \Gamma_{78}) \quad (2.1)$$

Note that $J^T = J$ and $J^2 = 2J + 3$. The matrix J is defined such that it is proportional to $F_{ab}\Gamma^{ab}$ where $F_{(2)}$ is the two-form field strength given in (A.15), $a, b = 0, \dots, 9$ being flat target space-time indices. $F_{(2)}$ is proportional to the Kähler form on \mathbb{CP}^3 . In terms of (2.1) the projector on to the supersymmetric fermionic directions is

$$P = \frac{3 - J}{4} \quad (2.2)$$

Thus all supersymmetric fermionic directions are characterized by $P\theta = \theta$ or equivalently $J\theta = -\theta$.

If $\Gamma_{5678}\theta = -\theta$ we see that $J\theta = -\theta$. Hence this gives 16 supersymmetric directions. If $\Gamma_{5678}\theta = \theta$ then we need in addition that $\Gamma_{4956}\theta = \theta$. Hence this gives 8 supersymmetric directions. Thus we see that we have in total 24 supersymmetric directions. The 8 remaining non-supersymmetric directions are characterized by $\Gamma_{5678}\theta = \theta$ and $\Gamma_{4956}\theta = -\theta$ corresponding to $J\theta = 3\theta$.

2.2 Supervielbeins and Lagrangian

We now present the GS Lagrangian for the 24 supersymmetric directions. Thus, we assume in the following that θ obey $P\theta = \theta$.

Write the world-sheet metric as s_{AB} with the world-sheet indices $A, B = 0, 1$. Then we define $h^{AB} = \sqrt{|\det s|} s^{AB}$. Thus $\det h = -1$. We furthermore define the epsilon symbol ε^{AB} such that $\varepsilon^{01} = \varepsilon_{01} = 1$.

Introduce for $0 \leq s \leq 1$ the supervielbeins

$$E(s)^a = e^a + 4i\bar{\theta}\Gamma^a \frac{\sinh^2(\frac{s}{2}\mathcal{M})}{\mathcal{M}^2} D\theta, \quad E(s)^\alpha = \left(\frac{\sinh s\mathcal{M}}{\mathcal{M}} D\theta \right)^\alpha \quad (2.3)$$

where a is the flat target space-time index. We write

$$E^a = E(s=1)^a = e^a + 4i\bar{\theta}\Gamma^a \frac{\sinh^2(\frac{1}{2}\mathcal{M})}{\mathcal{M}^2} D\theta, \quad E^\alpha = E(s=1)^\alpha = \left(\frac{\sinh \mathcal{M}}{\mathcal{M}} D\theta \right)^\alpha \quad (2.4)$$

The covariant derivative is

$$D\theta = P(d - \frac{1}{R}\Gamma_{0123}\Gamma_a e^a + \frac{1}{4}\omega^{ab}\Gamma_{ab})\theta \quad (2.5)$$

The two-fermion matrix \mathcal{M}^2 can be found in terms of the structure constants of the generators of $OSp(6|2, 2)$. Schematically we write

$$(\mathcal{M}^2)_\beta^\alpha = -\theta^\gamma \tilde{f}_{\gamma i}^\alpha \theta^\delta \hat{f}_{\delta\beta}^i \quad (2.6)$$

in terms of the structure constants of the $OSp(6|2, 2)$ algebra (C.1) given explicitly by (C.2)-(C.3) in Appendix C. By Eq. (C.9) in Appendix C we have

$$\begin{aligned} (\mathcal{M}^2)_\beta^\alpha &= -\frac{2i}{R}(P\Gamma_{0123}\Gamma_{\hat{a}}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma^{\hat{a}}P)_{\delta\beta} - \frac{i}{R}(P\Gamma_{0123}\Gamma_{a'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma^{a'}P)_{\delta\beta} \\ &\quad + \frac{i}{R}(P\Gamma_{11}\Gamma_{a'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma_{11}\Gamma^{a'}P)_{\delta\beta} - \frac{i}{R}(P\Gamma_{\hat{a}\hat{b}}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma^{\hat{a}\hat{b}}P)_{\delta\beta} \\ &\quad + \frac{i}{2R}(P\Gamma_{a'b'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma^{a'b'}P)_{\delta\beta} - \frac{i}{R}(\Gamma_{0123}\Gamma_{11})^\alpha_\gamma \theta^\gamma \theta^\delta (\Gamma^0\Gamma_{11})_{\delta\beta} \end{aligned} \quad (2.7)$$

where $\hat{a}, \hat{b} = 0, 1, 2, 3$ and $a', b' = 4, \dots, 9$. Note that $\mathcal{M}^2 = P\mathcal{M}^2P$. It is shown in Appendix C that this expression is equivalent to the one found in [22, 23] in a different representation of the $OSp(6|2, 2)$ algebra.

From the supervielbeins (2.3)-(2.4) we can construct the generalized Maurer-Cartan forms

$$L(s)_A^a = E(s)_\mu^a \partial_A X^\mu + E(s)_\alpha^a \partial_A \theta^\alpha, \quad L(s)_A^\alpha = E(s)_\mu^\alpha \partial_A X^\mu + E(s)_\beta^\alpha \partial_A \theta^\beta \quad (2.8)$$

for $0 \leq s \leq 1$. We define then the Maurer-Cartan forms $L_A^a = L(s=1)_A^a$ and $L_A^\alpha = L(s=1)_A^\alpha$. We can now write the type IIA GS Lagrangian for the supersymmetric fermionic directions on $\text{AdS}_4 \times \mathbb{CP}^3$ [22, 23], based on the supercoset construction of [45, 46], as

$$\mathcal{L} = -\frac{1}{2}h^{AB}\eta_{ab}L_A^a L_B^b - 2i\varepsilon^{AB} \int_0^1 ds L(s)_A^a (\bar{\theta}\Gamma_a\Gamma_{11})_\alpha L(s)_B^\alpha \quad (2.9)$$

The Virasoro constraints are

$$S_{AB} = \frac{1}{2} h_{AB} h^{CD} S_{CD} , \quad S_{AB} \equiv \eta_{ab} L_A^a L_B^b \quad (2.10)$$

From now on we shall truncate the Lagrangian to include terms with at most four fermions, since this is the order relevant to compute one-loop corrections to pp-wave energies. Dividing S_{AB} according to the number of fermions we have

$$S_{AB} = S_{AB}^{(0f)} + S_{AB}^{(2f)} + S_{AB}^{(4f)} \quad (2.11)$$

we compute

$$S_{AB}^{(0f)} = g_{\mu\nu} \partial_A X^\mu \partial_B X^\nu \quad (2.12)$$

$$S_{AB}^{(2f)} = i \bar{\theta} \Gamma_\mu (\partial_A X^\mu D_B \theta + \partial_B X^\mu D_A \theta) \quad (2.13)$$

$$S_{AB}^{(4f)} = -(\bar{\theta} \Gamma^a D_A \theta)(\bar{\theta} \Gamma_a D_B \theta) + \frac{i}{12} \bar{\theta} \Gamma_\mu \mathcal{M}^2 (\partial_A X^\mu D_B \theta + \partial_B X^\mu D_A \theta) \quad (2.14)$$

We write the total Lagrangian (2.9) as

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{WZ}} \quad (2.15)$$

where the kinetic part is

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2} h^{AB} S_{AB} \quad (2.16)$$

and the Wess-Zumino part is

$$\mathcal{L}_{\text{WZ}} = \mathcal{L}_{\text{WZ}}^{(2f)} + \mathcal{L}_{\text{WZ}}^{(4f)} \quad (2.17)$$

where

$$\mathcal{L}_{\text{WZ}}^{(2f)} = -i \varepsilon^{AB} \partial_A X^\mu \bar{\theta} \Gamma_\mu \Gamma_{11} D_B \theta \quad (2.18)$$

and

$$\mathcal{L}_{\text{WZ}}^{(4f)} = -\frac{i}{12} \varepsilon^{AB} \partial_A X^\mu \bar{\theta} \Gamma_\mu \Gamma_{11} \mathcal{M}^2 D_B \theta + \frac{1}{2} \varepsilon^{AB} (\bar{\theta} \Gamma^a D_A \theta)(\bar{\theta} \Gamma_a \Gamma_{11} D_B \theta) \quad (2.19)$$

3 Equivalence with general type IIA Lagrangian for two-fermion terms

In this section we show that the two-fermion terms in the type IIA superspace Lagrangian (2.9) for the $\text{AdS}_4 \times \mathbb{CP}^3$ background, restricting to the supersymmetric fermionic directions, are equivalent to those of the type IIA GS Lagrangian, found by Cvetič et al. [24] for general type IIA backgrounds, on this particular background. We use here the explicit expressions for the $\text{AdS}_4 \times \mathbb{CP}^3$ background written in Appendix A.²

²While this paper was in preparation the paper [47] appeared where the equivalence between (2.9) and the Lagrangian of [24] is also examined.

General GS superstring action for type IIA

The type IIA superstring Lagrangian including two-fermion terms for a background with zero Kalb-Ramond field and zero dilaton field can be written as for the superspace, as a sum of kinetic and Wess-Zumino part [24]

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{WZ}} \quad (3.1)$$

with the kinetic part given by

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}h^{AB}S_{AB} \quad (3.2)$$

$$S_{AB} = g_{\mu\nu}\partial_A X^\mu\partial_B X^\nu + i\bar{\theta}\Gamma_\mu(\partial_A X^\mu\tilde{D}_B\theta + \partial_B X^\mu\tilde{D}_A\theta) + \frac{i}{8}\partial_A X^\mu\partial_B X^\nu\bar{\theta}(\Gamma_\mu M\Gamma_\nu + \Gamma_\nu M\Gamma_\mu)\theta \quad (3.3)$$

and the Wess-Zumino part given by

$$\mathcal{L}_{\text{WZ}} = i\varepsilon^{AB}\bar{\theta}\Gamma_{11}\Gamma_\mu\partial_A X^\mu\tilde{D}_B\theta + \frac{i}{8}\varepsilon^{AB}\partial_A X^\mu\partial_B X^\nu\bar{\theta}\Gamma_{11}\Gamma_\mu M\Gamma_\nu\theta \quad (3.4)$$

where the matrix M is defined as

$$M = -\frac{1}{2}F_{\mu\nu}\Gamma_{11}\Gamma^{\mu\nu} + \frac{1}{24}F_{\mu\nu\rho\sigma}\Gamma^{\mu\nu\rho\sigma} \quad (3.5)$$

where $F_{(2)}$ and $F_{(4)}$ are the two and four-form Ramond-Ramond field strengths and the covariant derivative $\tilde{D}_A\theta$ is

$$\tilde{D}_A\theta = \partial_A\theta + \frac{1}{4}\partial_A X^\mu\omega_\mu^{ab}\Gamma_{ab}\theta \quad (3.6)$$

where ω_μ^{ab} is the spin-connection with a, b being flat indices. The Virasoro constraints are again given by

$$S_{AB} = \frac{1}{2}h_{AB}h^{CD}S_{CD} \quad (3.7)$$

Equivalence with superspace action

Using (A.15)-(A.16) we compute

$$M = -\frac{8}{R}\Gamma_{0123}P \quad (3.8)$$

Inserting this into (3.3) and (3.4), with covariant derivative (3.6), we see that Eqs. (3.3) and (3.4) are equivalent to Eqs. (2.13) and (2.18), with covariant derivative (3.7), provided we have that $\tilde{D}_A\theta = P\tilde{D}_A\theta$ for any spinor with $P\theta = \theta$ on the $\text{AdS}_4 \times \mathbb{CP}^3$ background. This is true if

$$\omega_\mu^{ab}[P, \Gamma_{ab}] = 0 \quad (3.9)$$

We can now check this using the spin connection as computed from the zehnbeins (A.11)-(A.14). For nearly all non-zero components of ω_μ^{ab} you have that a and b are such that $[P, \Gamma_{ab}] = 0$. The only 8 components for which this is not the case are $\omega_{x_1}^{45}, \omega_{x_1}^{69}, \omega_{y_1}^{46}, \omega_{y_1}^{59}, \omega_{x_2}^{47}, \omega_{x_2}^{89}, \omega_{y_2}^{48}$ and $\omega_{y_2}^{79}$. However, if we consider $\omega_{x_1}^{45}$ and $\omega_{x_1}^{69}$ we see that $\omega_{x_1}^{45} = -\omega_{x_1}^{69}$. For (3.9) to hold for $\mu = x_1$ it is therefore sufficient that $[P, \Gamma_{45} - \Gamma_{69}] = 0$ which indeed is the case, as one can check explicitly using (2.1)-(2.2). It works similar for $\mu = y_1, x_2, y_2$ hence we have checked

explicitly that Eq. (3.9) holds. Actually the group theoretical reason of (3.9) is as follows. The \mathbb{CP}^3 part of the spin connection $\omega^{ab}\Gamma_{ab}$ takes values in the algebra of the $SU(3) \times U(1)$ stability group of \mathbb{CP}^3 . The $U(1)$ subgroup of this stability group is generated by the Kähler form J which enters the projector P . This insures that P commutes with $\omega^{ab}\Gamma_{ab}$.

We can thus conclude that the two-fermion terms in the Lagrangian and Virasoro constraints (2.9)-(2.10) agree with the Lagrangian and Virasoro constraints (3.1)-(3.7) on the $\text{AdS}_4 \times \mathbb{CP}^3$ background for the supersymmetric fermionic directions.

4 Light-cone Lagrangian

In this section we impose the light-cone gauge for the type IIA GS Lagrangian (2.9) on $\text{AdS}_4 \times \mathbb{CP}^3$ restricted to the supersymmetric fermionic directions. We find explicit expressions for the full gauge-fixed light-cone Lagrangian for terms quadratic, cubic and quartic in the number of fields, expanding in powers of $1/R$ around the pp-wave background where R is the radius of \mathbb{CP}^3 in the $\text{AdS}_4 \times \mathbb{CP}^3$ background (see Appendix A). The gauge fixed Lagrangian found in this Section is then simplified using the fermionic field redefinition in Section 6.

4.1 Outline of general procedure

We consider the $\text{AdS}_4 \times \mathbb{CP}^3$ background described in Appendix A. We examine string excitations around a null curve $\delta = t/2$ at $u_i = 0$, $i = 1, 2, 3, 4$ and $y_1 = y_2 = 0$ in the limit of $R \rightarrow \infty$. More specifically, we take the near plane wave limit $R \rightarrow \infty$, keeping the coordinates t, v, u_i, x_a, y_a fixed, $i = 1, 2, 3, 4$ and $a = 1, 2$.

In the following we wish to consider the Lagrangian in the light-cone gauge

$$t(\tau, \sigma) = c\tau \quad (4.1)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{v}} = \text{constant} , \quad \frac{\partial \mathcal{L}}{\partial v'} = 0 \quad (4.2)$$

For choices of the light-cone directions lying both in the AdS, see [48]. We allow for corrections of the world-sheet metric

$$h^{\tau\tau} = -1 + \frac{q_1}{R} + \frac{q_2}{R^2} + \mathcal{O}(R^{-3}) , \quad h^{\tau\sigma} = \frac{q_3}{R} + \frac{q_4}{R^2} + \mathcal{O}(R^{-3}) \quad (4.3)$$

The procedure is now to construct S_{AB} and \mathcal{L}_{WZ} from the expressions of Section 2. Using S_{AB} with (4.1) imposed we can write down the two independent Virasoro constraints. These two constraints can be solved for \dot{v} and v' , order by order in $1/R$.

Inserting \dot{v} and v' into the two gauge conditions (4.2) we can solve for the corrections to the world-sheet metric (4.3).

Inserting now both \dot{v} and v' and the corrections to the world-sheet metric in the following expression

$$\mathcal{L}_{\text{gf}} = \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{v}} \dot{v} \quad (4.4)$$

we obtain the gauge fixed Lagrangian \mathcal{L}_{gf} . We write the expanded gauge fixed Lagrangian as

$$\mathcal{L}_{\text{gf}} = \mathcal{L}_{2,B} + \mathcal{L}_{2,F} + \frac{1}{R}(\mathcal{L}_{3,B} + \mathcal{L}_{3,BF}) + \frac{1}{R^2}(\mathcal{L}_{4,B} + \mathcal{L}_{4,BF} + \mathcal{L}_{4,F}) + \mathcal{O}(R^{-3}) \quad (4.5)$$

4.2 Bosonic terms in Lagrangian

Employing the procedure explained above it is straightforward to compute the bosonic part of the Lagrangian

$$\mathcal{L}_{2,B} = \frac{1}{2} \sum_{i=1}^4 (\dot{u}_i^2 - u_i'^2 - c^2 u_i^2) + \frac{1}{16} \sum_{a=1}^2 (\dot{x}_a^2 - x_a'^2 + 2c y_a \dot{x}_a + \dot{y}_a^2 - y_a'^2) \quad (4.6)$$

$$\mathcal{L}_{3,B} = \frac{u_4}{8} (x_1'^2 + y_1'^2 - x_2'^2 - y_2'^2 - \dot{x}_1^2 - \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) \quad (4.7)$$

$$\begin{aligned} \mathcal{L}_{4,B} = & -\frac{2}{c^2} \left(\sum_{i=1}^8 \dot{X}^i X'^i \right)^2 - c^2 \left(\sum_{i=1}^3 u_i^2 \right)^2 + \frac{2}{3} c^2 u_4^4 + \sum_{i,j=1}^3 u_i^2 (\dot{u}_j^2 - u_j'^2) \\ & + \frac{1}{2c^2} \left(c^2 \sum_{i=1}^4 u_i^2 - \sum_{i=1}^8 [(\dot{X}^i)^2 + (X'^i)^2] \right) \left(c^2 \sum_{j=1}^3 u_j^2 - 3c^2 u_4^2 - \sum_{j=1}^8 [(\dot{X}^j)^2 + (X'^j)^2] \right) \\ & - \frac{c}{48} \sum_{a=1}^2 \dot{x}_a y_a^3 + \frac{1}{16} \sum_{a=1}^2 y_a^2 (x_a'^2 - \dot{x}_a^2) \end{aligned} \quad (4.8)$$

where we have introduced the compact notation

$$\begin{aligned} \dot{X}^{i=1\dots 4} &= (\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4) , \quad \dot{X}^{i=5\dots 8} = \frac{\sqrt{2}}{4} (\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2) \\ X'^{i=1\dots 4} &= (u_1', u_2', u_3', u_4') , \quad X'^{i=5\dots 8} = \frac{\sqrt{2}}{4} (x_1', y_1', x_2', y_2') \end{aligned} \quad (4.9)$$

4.3 Introduction of fermionic terms

We define the purely fermionic bilinears

$$\begin{aligned} A_{a,A} &= \bar{\theta} \Gamma_a \partial_A \theta , \quad \tilde{A}_{a,A} = \bar{\theta} \Gamma_{11} \Gamma_a \partial_A \theta \\ B_{abc} &= \bar{\theta} \Gamma_a \Gamma_{bc} \theta , \quad \tilde{B}_{abc} = \bar{\theta} \Gamma_{11} \Gamma_a \Gamma_{bc} \theta \\ C_{ab} &= \bar{\theta} \Gamma_a P \Gamma_{0123} \Gamma_b \theta , \quad \tilde{C}_{ab} = \bar{\theta} \Gamma_{11} \Gamma_a P \Gamma_{0123} \Gamma_b \theta \end{aligned} \quad (4.10)$$

Note the following important properties

$$\begin{aligned} C_{ab} &= C_{ba} , \quad \tilde{C}_{ab} = -\tilde{C}_{ba} \\ B_{abc} &= \bar{\theta} \Gamma_{abc} \theta , \quad \tilde{B}_{abc} = \bar{\theta} \Gamma_{11} \Gamma_{abc} \theta \end{aligned} \quad (4.11)$$

which can be deduced using that $(\Gamma^0 \Gamma_a)^T = \Gamma^0 \Gamma_a$. In terms of the fermionic bilinears (4.10) the quantities appearing in the Lagrangian and Virasoro constraints are

$$\begin{aligned} S_{AB}^{(2f)} &= i A_{a,B} e_\mu^a \partial_A X^\mu + i A_{a,A} e_\mu^a \partial_B X^\mu + \frac{i}{4} B_{abc} e_\mu^a \omega_\nu^{bc} (\partial_A X^\mu \partial_B X^\nu + \partial_B X^\mu \partial_A X^\nu) \\ &\quad - \frac{2i}{R} C_{ab} e_\mu^a e_\nu^b \partial_A X^\mu \partial_B X^\nu \end{aligned} \quad (4.12)$$

$$\mathcal{L}_{\text{WZ}}^{(2f)} = i \varepsilon^{AB} \tilde{A}_{a,B} e_\mu^a \partial_A X^\mu + \frac{i}{4} \varepsilon^{AB} \tilde{B}_{abc} e_\mu^a \omega_\nu^{bc} \partial_A X^\mu \partial_B X^\nu - \frac{i}{R} \varepsilon^{AB} \tilde{C}_{ab} e_\mu^a e_\nu^b \partial_A X^\mu \partial_B X^\nu \quad (4.13)$$

Comparing with Eqs. (2.13) and (2.18) we note that we removed the projector P in our definition of B_{abc} and \tilde{B}_{abc} in (4.10). This is allowed since (3.9) holds for the $\text{AdS}_4 \times \mathbb{CP}^3$ background.

From (2.14) and (2.19) we see that the four-fermion terms are

$$S_{00}^{(4f)} = -\eta^{ab} \left(A_{a,\tau} - \frac{c}{2} C_{+a} + \frac{c}{4} (B_{a56} + B_{a78}) \right) \left(A_{b,\tau} - \frac{c}{2} C_{+b} + \frac{c}{4} (B_{b56} + B_{b78}) \right) \\ + \frac{ic}{6} \bar{\theta} \Gamma_+ \mathcal{M}^2 \dot{\theta} - \frac{ic^2}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 P \Gamma_{0123} \Gamma_+ \theta + \frac{ic^2}{24} \bar{\theta} \Gamma_+ \mathcal{M}^2 (\Gamma_{56} + \Gamma_{78}) \theta \quad (4.14)$$

$$S_{01}^{(4f)} = -\eta^{ab} \left(A_{a,\tau} - \frac{c}{2} C_{+a} + \frac{c}{4} (B_{a56} + B_{a78}) \right) A_{b,\sigma} + \frac{ic}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 \theta' \quad (4.15)$$

$$S_{11}^{(4f)} = -\eta^{ab} A_{a,\sigma} A_{b,\sigma} \quad (4.16)$$

$$\mathcal{L}_{\text{WZ}}^{(4f)} = \frac{ic}{12} \bar{\theta} \Gamma_{11} \Gamma_+ \mathcal{M}^2 \theta' - \frac{1}{2} \eta^{ab} \left[A_{a,\tau} - \frac{c}{2} C_{+a} + \frac{c}{4} (B_{a56} + B_{a78}) \right] \tilde{A}_{b,\sigma} \\ + \frac{1}{2} \eta^{ab} \left[\tilde{A}_{a,\tau} + \frac{c}{2} \tilde{C}_{+a} + \frac{c}{4} (\tilde{B}_{a56} + \tilde{B}_{a78}) \right] A_{b,\sigma} \quad (4.17)$$

where we used that

$$\bar{\theta} \Gamma_a D_\tau \theta = A_{a,\tau} - \frac{c}{2} C_{+a} + \frac{c}{4} (B_{a56} + B_{a78}) + \mathcal{O}(R^{-1}) \\ \bar{\theta} \Gamma_{11} \Gamma_a D_\tau \theta = \tilde{A}_{a,\tau} + \frac{c}{2} \tilde{C}_{+a} + \frac{c}{4} (\tilde{B}_{a56} + \tilde{B}_{a78}) + \mathcal{O}(R^{-1}) \quad (4.18)$$

4.4 Fixing κ -symmetry

We are considering here the κ -symmetry transformation on the supersymmetric fermionic directions. We have already imposed a partial κ -symmetry gauge choice by demanding $P\theta = \theta$ thus reducing the number of fermionic directions from 32 to 24. In the following we fix the remaining 8 directions in the κ -symmetry by a gauge choice that follows from our light-cone gauge.

In general the κ -symmetry variations are (assuming we are on the space of supersymmetric directions $P\theta = \theta$)

$$E_\mu^\alpha \delta X^\mu + E_\beta^\alpha \delta \theta^\beta = [(1 + \Gamma)\kappa]^\alpha \quad (4.19)$$

$$E_\mu^a \delta X^\mu + E_\alpha^a \delta \theta^\alpha = 0 \quad (4.20)$$

where $(1 + \Gamma)/2$ is a spinor projection matrix defined in [49, 50, 51].

We analyze the κ -symmetry in the Penrose limit. In this limit we have the super vielbeins

$$E^a = e^a + i\bar{\theta} \Gamma^a D\theta, \quad E^\alpha = (D\theta)^\alpha \quad (4.21)$$

Eqs. (4.19)-(4.20) give

$$\delta X^\mu (D_\mu \theta)^\alpha + \delta \theta^\alpha = [(1 + \Gamma)\kappa]^\alpha, \quad \delta t + 2i\bar{\theta} \Gamma^+ D_\mu \theta \delta X^\mu + 2i\bar{\theta} \Gamma^+ \delta \theta = 0 \quad (4.22)$$

using $e^+ = \frac{1}{2}dt$ and assuming $P\theta = \theta$ and $P\delta\theta = \delta\theta$. Combining these equations we get

$$\delta t = -2i\bar{\theta}\Gamma^+(1+\Gamma)\kappa \quad (4.23)$$

For the light-cone gauge to be consistent we need that $\delta t = 0$ under variations of κ -symmetry. Suppose now we have a supersymmetric fermionic direction with $P\Gamma^-P\theta = 0$. Then we see from (4.23) that $\delta t = 2i\theta^T(1+\Gamma)\kappa$. Thus, such fermionic directions are clearly not consistent with the light-cone gauge. The matrix $P\Gamma^-P$ has 8 supersymmetric fermionic directions with eigenvalue zero, characterized by $\Gamma_{5678}\theta = -\theta$ and $\Gamma^-\theta = 0$. We fix the remaining κ -symmetry gauge freedom by demanding that these directions are put to zero. This leaves the following 16 physical fermionic directions in the light cone gauge

$$\begin{aligned} & 8 \text{ fermionic directions defined by } \Gamma_{5678}\theta = -\theta, \Gamma^+\theta = 0 \\ & 8 \text{ fermionic directions defined by } \Gamma_{5678}\theta = \theta, \Gamma_{4956}\theta = \theta \end{aligned} \quad (4.24)$$

It is useful to parameterize this by introducing the projectors

$$\begin{aligned} \mathcal{P}_+ &= \frac{I + \Gamma_{5678}}{2} \frac{I + \Gamma_{4956}}{2}, \quad \mathcal{P}_- = \frac{I - \Gamma_{5678}}{2} \frac{I - \Gamma_{09}}{2} \\ \mathcal{P}'_+ &= \frac{I + \Gamma_{5678}}{2} \frac{I - \Gamma_{4956}}{2}, \quad \mathcal{P}'_- = \frac{I - \Gamma_{5678}}{2} \frac{I + \Gamma_{09}}{2} \end{aligned} \quad (4.25)$$

named after the eigenvalue of Γ_{5678} . These projectors are mutually orthogonal to each other and they are all idempotent and symmetric. We have

$$P = \mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}'_- , \quad I = \mathcal{P}_+ + \mathcal{P}_- + \mathcal{P}'_+ + \mathcal{P}'_- \quad (4.26)$$

We are thus imposing that our spinor $\theta = \theta^1 + \theta^2$ obeys

$$(\mathcal{P}_+ + \mathcal{P}_-)\theta = \theta \quad (4.27)$$

or, equivalently, $(\mathcal{P}'_+ + \mathcal{P}'_-)\theta = 0$. This is our condition for physical fermionic modes.

It is worth noting that the gauge condition (4.24) is different from the κ -symmetry gauge fixing condition that one imposes for string theory in $AdS_5 \times S^5$. As it is well known, in that case it is sufficient to impose the condition $\Gamma^+\theta = 0$ for all the fermionic directions, while in this case, due to the less amount of supersymmetry, the condition $\Gamma^+\theta = 0$ does not only select supersymmetric states.

4.5 Explicit fermionic terms in Lagrangian

We compute here the fermionic quantities appearing in (4.5) in terms of the fermionic bilinears defined in (4.10)

$$\mathcal{L}_{2,F} = \frac{ic}{2}A_{+,\tau} + \frac{ic}{2}\tilde{A}_{+,\sigma} + \frac{ic^2}{8}(B_{+56} + B_{+78}) - \frac{ic^2}{4}C_{++} \quad (4.28)$$

$$\begin{aligned}
\mathcal{L}_{3,BF} = & -ic \sum_{i=1}^8 (\tilde{C}_{+i} X'^i + C_{+i} \dot{X}^i) + i \sum_{i=1}^8 \left[(A_{i,\tau} + \tilde{A}_{i,\sigma}) \dot{X}^i - (\tilde{A}_{i,\tau} + A_{i,\sigma}) X'^i \right] \\
& + \frac{ic}{4} \sum_{i=1}^8 \left[(B_{i56} + B_{i78}) \dot{X}^i - (\tilde{B}_{i56} + \tilde{B}_{i78}) X'^i \right] + \frac{ic^2}{4} (B_{+56} - B_{+78}) u_4 \\
& + \frac{ic^2}{4} \sum_{i=1}^4 B_{+-i} u_i + \frac{ic}{4} \sum_{i=5}^8 s_i (B_{+4i} \dot{X}^i + \tilde{B}_{+4i} X'^i) + \frac{ic}{8} \sum_{i,j=5}^8 \epsilon_{ij} (B_{+-i} \dot{X}^j + \tilde{B}_{+-i} X'^j)
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\mathcal{L}_{4,BF} = & -iu_4 \sum_{i=5}^8 s_i \left[(A_{i,\tau} + \tilde{A}_{i,\sigma}) \dot{X}^i - (\tilde{A}_{i,\tau} + A_{i,\sigma}) X'^i \right] + icu_4^2 A_{+,\tau} + ic(u_1^2 + u_2^2 + u_3^2) \tilde{A}_{+,\sigma} \\
& + \frac{i}{c} \sum_{i=1}^8 \dot{X}^i X'^i (\tilde{A}_{+,\tau} - \tilde{A}_{-,\tau} - A_{+,\sigma} - A_{-,\sigma}) + \frac{i}{2c} \sum_{i=1}^8 ((\dot{X}^i)^2 + (X'^i)^2) (A_{+,\tau} + A_{-,\tau} - \tilde{A}_{+,\sigma} + \tilde{A}_{-,\sigma}) \\
& + i \sum_{i,j=1}^8 \left[C_{ij} (X'^i X'^j - \dot{X}^i \dot{X}^j) + 2\tilde{C}_{ij} X'^i \dot{X}^j \right] + icu_4 \sum_{i=5}^8 s_i (C_{+i} \dot{X}^i + \tilde{C}_{+i} X'^i) \\
& - i \sum_{i=1}^8 \dot{X}^i X'^i \tilde{C}_{+-} - \frac{ic^2}{2} \sum_{i=1}^4 u_i^2 C_{++} - \frac{i}{2} \sum_{i=1}^8 ((\dot{X}^i)^2 + (X'^i)^2) C_{+-} + ic \sum_{i,j=1}^3 u'_i u_j \tilde{B}_{+ij} \\
& + \frac{ic}{2} u_4 \sum_{i=1}^8 \left[(B_{i56} - B_{i78}) \dot{X}^i - (\tilde{B}_{i56} - \tilde{B}_{i78}) X'^i \right] - \frac{ic}{4} u_4 \sum_{i=5}^8 s_i \left[(B_{i56} + B_{i78}) \dot{X}^i - (\tilde{B}_{i56} + \tilde{B}_{i78}) X'^i \right] \\
& - \frac{ic}{2} \sum_{i=1}^8 \sum_{j=1}^4 u_j (B_{-ij} \dot{X}^i - \tilde{B}_{-ij} X'^i) - \frac{i}{2} \sum_{i=1}^8 \sum_{j=5}^8 s_j \left[(\dot{X}^i \dot{X}^j - X'^i X'^j) B_{4ij} + (\dot{X}^i X'^j - X'^i \dot{X}^j) \tilde{B}_{4ij} \right] \\
& + \frac{ic}{2} \sum_{i=1}^3 \sum_{j=4}^8 u_i \left[B_{+ij} \dot{X}^j - \tilde{B}_{+ij} X'^j \right] + \frac{i}{8} \sum_{i=1}^8 ((\dot{X}^i)^2 + (X'^i)^2) (B_{-56} + B_{-78} - B_{+56} - B_{+78}) \\
& + \frac{i}{4} \sum_{i=1}^8 \sum_{j,k=5}^8 \epsilon_{jk} \left[(B_{+ij} - B_{-ij}) (\dot{X}^i \dot{X}^k - X'^i X'^k) + (\tilde{B}_{+ij} - \tilde{B}_{-ij}) (\dot{X}^i X'^k - X'^i \dot{X}^k) \right] \\
& - \frac{ic}{8} u_4 \sum_{i,j=5}^8 s_i \epsilon_{ij} (B_{+i} \dot{X}^j + \tilde{B}_{+i} X'^j) + \frac{ic}{2} u_4 \sum_{i=1}^3 (B_{+i4} \dot{u}_i - \tilde{B}_{+i4} u'_i) \\
& - \frac{i}{4} \sum_{i=1}^8 \dot{X}^i X'^i (\tilde{B}_{+56} + \tilde{B}_{+78} + \tilde{B}_{-56} + \tilde{B}_{-78}) - \frac{ic}{4} (B_{+56} \dot{x}_1 y_1 + \tilde{B}_{+56} x'_1 y_1 + B_{+78} \dot{x}_2 y_2 + \tilde{B}_{+78} x'_2 y_2) \\
& + \frac{ic}{4} u_4 \sum_{i=5}^8 (-B_{+4i} \dot{X}^i + 3\tilde{B}_{+4i} X'^i) + \frac{ic^2}{4} (B_{+56} + B_{+78}) (u_1^2 + u_2^2 + u_3^2 + 2u_4^2)
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\mathcal{L}_{4,F} = & \frac{ic}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 \dot{\theta} - \frac{ic^2}{24} \bar{\theta} \Gamma_+ \mathcal{M}^2 P \Gamma_{0123} \Gamma_+ \theta + \frac{ic^2}{48} \bar{\theta} \Gamma_+ \mathcal{M}^2 (\Gamma_{56} + \Gamma_{78}) \theta + \frac{ic}{12} \bar{\theta} \Gamma_{11} \Gamma_+ \mathcal{M}^2 \theta' \\
& - \frac{1}{2} \sum_{i=1}^8 \left[A_{i,\tau} - \frac{c}{2} C_{+i} + \frac{c}{4} (B_{i56} + B_{i78}) \right]^2 + \frac{1}{2} \sum_{i=1}^8 A_{i,\sigma}^2 \\
& - \frac{1}{2} \sum_{i=1}^8 \tilde{A}_{i,\sigma} \left[A_{i,\tau} - \frac{c}{2} C_{+i} + \frac{c}{4} (B_{i56} + B_{i78}) \right] + \frac{1}{2} \sum_{i=1}^8 A_{i,\sigma} \left[\tilde{A}_{i,\tau} + \frac{c}{2} \tilde{C}_{+i} + \frac{c}{4} (\tilde{B}_{i56} + \tilde{B}_{i78}) \right] \\
& - \frac{1}{2} A_{+,\sigma} \left[\tilde{A}_{+,\tau} - \frac{1}{2} \tilde{A}_{-,\tau} - \frac{3c}{4} \tilde{C}_{+-} - \frac{c}{4} (\tilde{B}_{+56} + \tilde{B}_{+78}) - \frac{c}{8} (\tilde{B}_{-56} + \tilde{B}_{-78}) \right] \\
& + \frac{1}{2} \tilde{A}_{+,\sigma} \left[A_{+,\tau} + \frac{1}{2} A_{-,\tau} - \frac{c}{2} C_{++} - \frac{c}{4} C_{+-} + \frac{c}{4} (B_{+56} + B_{+78}) + \frac{c}{8} (B_{-56} + B_{-78}) \right] \\
& - \frac{1}{4} A_{-,\sigma} \left[\tilde{A}_{+,\tau} + \frac{c}{4} (\tilde{B}_{+56} + \tilde{B}_{+78}) \right] - \frac{1}{4} \tilde{A}_{-,\sigma} \left[A_{+,\tau} - \frac{c}{2} C_{++} + \frac{c}{4} (B_{+56} + B_{+78}) \right] + \\
& + \frac{c}{4} \left[A_{+,\tau} - \frac{c}{2} C_{++} + \frac{c}{4} (B_{+56} + B_{+78}) \right] (C_{+-} - C_{++} + B_{+56} + B_{+78})
\end{aligned} \tag{4.31}$$

where we used the notation defined in (4.9) and

$$s_5 = s_6 = 1, \quad s_7 = s_8 = -1, \quad \epsilon_{56} = -\epsilon_{65} = \epsilon_{78} = -\epsilon_{87} = 1 \tag{4.32}$$

with all other entries of ϵ_{ij} being zero.

5 PP-wave Lagrangian, Hamiltonian and field expansions

We analyze in this section in detail the pp-wave Lagrangian \mathcal{L}_2 and pp-wave Hamiltonian \mathcal{H}_2 emerging in the $R \rightarrow \infty$ limit. We split up the Lagrangian and Hamiltonian in the bosonic and fermionic parts

$$\mathcal{L}_2 = \mathcal{L}_{2,B} + \mathcal{L}_{2,F}, \quad \mathcal{H}_2 = \mathcal{H}_{2,B} + \mathcal{H}_{2,F} \tag{5.1}$$

and analyze these separately in the following. The pp-wave spectrum on this background was derived also in [27, 28, 29, 5, 8]. The Penrose limit used here was found in [8].³

5.1 Bosonic part

The quadratic bosonic Lagrangian is

$$\mathcal{L}_{2,B} = \frac{1}{2} \sum_{i=1}^4 (\dot{u}_i^2 - u_i'^2 - c^2 u_i^2) + \frac{1}{16} \sum_{a=1}^2 (\dot{x}_a^2 - x_a'^2 + 2c y_a \dot{x}_a + \dot{y}_a^2 - y_a'^2) \tag{5.2}$$

The momentum conjugate fields are defined by

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{5.3}$$

³Notice that the resulting pp-wave background has two flat directions which makes it particularly well suited for studying the $SU(2) \times SU(2)$ sector [8]. The coordinate system of this pp-wave is similar to the one found by a Penrose limit in [52] which is particularly well suited to study the $SU(2)$ sector of $\text{AdS}_5 \times S^5$, as explained in [53, 54]. See also [55].

We get $\Pi_{x_a} = (\dot{x}_a + cy_a)/8$, $\Pi_{y_a} = \dot{y}_a/8$ and $\Pi_{u_i} = \dot{u}_i$. The quadratic bosonic Hamiltonian is

$$c\mathcal{H}_{2,B} = \frac{1}{16} \sum_{a=1}^2 \left[p_{x_a}^2 + p_{y_a}^2 + x_a'^2 + y_a'^2 \right] + \frac{1}{2} \sum_{i=1}^4 \left[p_{u_i}^2 + u_i'^2 + c^2 u_i^2 \right] \quad (5.4)$$

where for convenience we defined the fields

$$p_{x_a} \equiv 8\Pi_{x_a} - cy_a, \quad p_{y_a} \equiv 8\Pi_{y_a}, \quad p_{u_i} \equiv \Pi_{u_i} \quad (5.5)$$

Notice that these fields are functions of the momenta and position variables.

The mode expansion for the bosonic fields can be written as

$$z_a(\tau, \sigma) = 2\sqrt{2} e^{i\frac{c\tau}{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\omega_n}} \left[a_n^a e^{-i(\omega_n \tau - n\sigma)} - (\tilde{a}_n^a)^\dagger e^{i(\omega_n \tau - n\sigma)} \right] \quad (5.6)$$

$$u_i(\tau, \sigma) = i \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{\Omega_n}} \left[\hat{a}_n^i e^{-i(\Omega_n \tau - n\sigma)} - (\hat{a}_n^i)^\dagger e^{i(\Omega_n \tau - n\sigma)} \right] \quad (5.7)$$

where

$$\omega_n = \sqrt{\frac{c^2}{4} + n^2}, \quad \Omega_n = \sqrt{c^2 + n^2} \quad (5.8)$$

and we defined $z_a(\tau, \sigma) = x_a(\tau, \sigma) + iy_a(\tau, \sigma)$. The canonical commutation relations $[x_a(\tau, \sigma), \Pi_{x_b}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma')$, $[y_a(\tau, \sigma), \Pi_{y_b}(\tau, \sigma')] = i\delta_{ab}\delta(\sigma - \sigma')$ and $[u_i(\tau, \sigma), \Pi_{u_j}(\tau, \sigma')] = i\delta_{ij}\delta(\sigma - \sigma')$ follow from

$$[a_m^a, (a_n^b)^\dagger] = \delta_{mn}\delta_{ab}, \quad [\tilde{a}_m^a, (\tilde{a}_n^b)^\dagger] = \delta_{mn}\delta_{ab}, \quad [\hat{a}_m^i, (\hat{a}_n^j)^\dagger] = \delta_{mn}\delta_{ij} \quad (5.9)$$

Employing the previous relations we obtain the bosonic free spectrum as

$$cH_2 = \sum_{i=1}^4 \sum_{n \in \mathbb{Z}} \Omega_n \hat{N}_n^i + \sum_{a=1}^2 \sum_{n \in \mathbb{Z}} \left(\omega_n - \frac{c}{2} \right) M_n^a + \sum_{a=1}^2 \sum_{n \in \mathbb{Z}} \left(\omega_n + \frac{c}{2} \right) N_n^a \quad (5.10)$$

with the number operators $\hat{N}_n^i = (\hat{a}_n^i)^\dagger \hat{a}_n^i$, $M_n^a = (a_n^a)^\dagger a_n^a$ and $N_n^a = (\tilde{a}_n^a)^\dagger \tilde{a}_n^a$, and with the level-matching condition

$$\sum_{n \in \mathbb{Z}} n \left[\sum_{i=1}^4 \hat{N}_n^i + \sum_{a=1}^2 (M_n^a + N_n^a) \right] = 0 \quad (5.11)$$

A peculiarity of the bosonic spectrum, as well as of the fermionic one, is that it contains four light and four heavy modes [29, 8]. The role of the heavy modes has been investigated in [41, 31].

5.2 Fermionic part

We now consider the fermionic part of the pp-wave Lagrangian. From (4.28) we have the quadratic fermionic Lagrangian

$$\mathcal{L}_{2,F} = \frac{ic}{2} A_{+,\tau} + \frac{ic}{2} \tilde{A}_{+,\sigma} + \frac{ic^2}{8} (B_{+56} + B_{+78}) - \frac{ic^2}{4} C_{++} \quad (5.12)$$

Using the identities

$$\mathcal{P}_+\Gamma_{09} = \Gamma_{09}\mathcal{P}'_+, \quad \mathcal{P}'_+\Gamma_{09} = \Gamma_{09}\mathcal{P}_+, \quad \mathcal{P}_-\Gamma_{09} = \Gamma_{09}\mathcal{P}_- = -\mathcal{P}_- \quad (5.13)$$

we compute

$$\begin{aligned} (\mathcal{P}_+ + \mathcal{P}_-)\Gamma^0\Gamma_+(\mathcal{P}_+ + \mathcal{P}_-) &= \mathcal{P}_+ + 2\mathcal{P}_- \\ (\mathcal{P}_+ + \mathcal{P}_-)\Gamma^0\Gamma_+P\Gamma_{0123}\Gamma_+(\mathcal{P}_+ + \mathcal{P}_-) &= (\mathcal{P}_+ + 4\mathcal{P}_-)\Gamma_{123} \end{aligned} \quad (5.14)$$

We can thus write the four terms in (5.12) as

$$\begin{aligned} A_{+,\tau} &= \theta(\mathcal{P}_+ + 2\mathcal{P}_-)\dot{\theta}, \quad \tilde{A}_{+,\sigma} = -\theta(\mathcal{P}_+ + 2\mathcal{P}_-)\Gamma_{11}\theta' \\ B_{+56} + B_{+78} &= \theta(\mathcal{P}_+ + 2\mathcal{P}_-)(\Gamma_{56} + \Gamma_{78})\theta, \quad C_{++} = \theta(\mathcal{P}_+ + 4\mathcal{P}_-)\Gamma_{123}\theta \end{aligned} \quad (5.15)$$

where we used our κ -symmetry gauge choice $(\mathcal{P}_+ + \mathcal{P}_-)\theta = \theta$.

Instead of parameterizing the fermionic directions in terms of the 32 component real spinor $\theta = \theta^1 + \theta^2$, $\Gamma_{11}\theta^1 = \theta^1$ and $\Gamma_{11}\theta^2 = -\theta^2$, we parameterize the fermionic directions in terms of the complex spinors

$$\psi = \theta^1 + i\Gamma_{049}\theta^2, \quad \psi^* = \theta^1 - i\Gamma_{049}\theta^2 \quad (5.16)$$

We see that $\Gamma_{11}\psi = \psi$ hence ψ has 16 complex components. From our choice of κ -symmetry gauge we have that physical spinors obey $(\mathcal{P}_+ + \mathcal{P}_-)\theta = \theta$ with the projectors \mathcal{P}_\pm defined by (4.25). Since Γ_{049} commutes with \mathcal{P}_\pm we get that the physical spinors ψ obey the condition

$$(\mathcal{P}_+ + \mathcal{P}_-)\psi = \psi \quad (5.17)$$

In the following we split up the spinor as $\psi = \psi_+ + \psi_-$ with $\psi_\pm = \mathcal{P}_\pm\psi$.

Using the above formulas in (5.12) we get the Lagrangian

$$\begin{aligned} \mathcal{L}_{2,F} &= \frac{ic}{2} \left[\psi_+^* \dot{\psi}_+ + 2\psi_-^* \dot{\psi}_- - \frac{1}{2} (\psi_+ \psi'_+ + \psi_+^* \psi'^*_+ + 2\psi_- \psi'_- + 2\psi_-^* \psi'^*_-) \right] \\ &\quad + \frac{c^2}{4} \psi_+ \psi_+^* - c^2 \psi_- \psi_-^* + \frac{ic^2}{2} \psi_- \Gamma_{56} \psi_-^* \end{aligned} \quad (5.18)$$

in terms of the physical spinor fields ψ_\pm . Note that we added here the total derivative

$$\frac{ic}{4} \partial_\tau (\psi_+^* \psi_+ + 2\psi_-^* \psi_-) \quad (5.19)$$

such that there is no $\dot{\psi}^*$ dependence in the Lagrangian.

From the corresponding e.o.m. we get the following mode expansions

$$\psi_{+,\alpha} = \frac{\sqrt{2\alpha'}}{\sqrt{c}} \sum_{n \in \mathbb{Z}} \left[f_n^+ d_{n,\alpha} e^{-i(\omega_n \tau - n\sigma)} - f_n^- d_{n,\alpha}^\dagger e^{i(\omega_n \tau - n\sigma)} \right] \quad (5.20)$$

$$\psi_{-,\alpha} = \frac{\sqrt{\alpha'}}{\sqrt{c}} (e^{-\frac{c}{2}\Gamma_{56}\tau})_{\alpha\beta} \sum_{n \in \mathbb{Z}} \left[-g_n^- b_{n,\beta} e^{-i(\Omega_n \tau - n\sigma)} + g_n^+ b_{n,\beta}^\dagger e^{i(\Omega_n \tau - n\sigma)} \right] \quad (5.21)$$

with the constants f_n^\pm and g_n^\pm defined by

$$f_n^\pm = \frac{\sqrt{\omega_n + n} \pm \sqrt{\omega_n - n}}{2\sqrt{\omega_n}}, \quad g_n^\pm = \frac{\sqrt{\Omega_n + n} \pm \sqrt{\Omega_n - n}}{2\sqrt{\Omega_n}} \quad (5.22)$$

and where the oscillators are subject to the conditions

$$\mathcal{P}_+ d_n = d_n, \quad \Gamma_{11} d_n = d_n, \quad \mathcal{P}_- b_n = b_n, \quad \Gamma_{11} b_n = b_n \quad (5.23)$$

and obey the anti-commutation relations

$$\{d_{m,\alpha}, d_{n,\beta}^\dagger\} = \delta_{mn} \left(\frac{1+\Gamma_{11}}{2} \mathcal{P}_+ \right)_{\alpha\beta}, \quad \{b_{m,\alpha}, b_{n,\beta}^\dagger\} = \delta_{mn} \left(\frac{1+\Gamma_{11}}{2} \mathcal{P}_- \right)_{\alpha\beta} \quad (5.24)$$

From this we obtain

$$\{\psi_{+,\alpha}(\tau, \sigma), \psi_{+,\beta}^*(\tau, \sigma')\} = \frac{4\pi\alpha'}{c} \left(\frac{1+\Gamma_{11}}{2} \mathcal{P}_+ \right)_{\alpha\beta} \delta(\sigma - \sigma') \quad (5.25)$$

$$\{\psi_{-,\alpha}(\tau, \sigma), \psi_{-,\beta}^*(\tau, \sigma')\} = \frac{2\pi\alpha'}{c} \left(\frac{1+\Gamma_{11}}{2} \mathcal{P}_- \right)_{\alpha\beta} \delta(\sigma - \sigma') \quad (5.26)$$

Hence

$$\{\psi_\alpha(\tau, \sigma), \psi_\beta^*(\tau, \sigma')\} = \frac{2\pi\alpha'}{c} \left(\frac{1+\Gamma_{11}}{2} (\mathcal{P}_- + 2\mathcal{P}_+) \right)_{\alpha\beta} \delta(\sigma - \sigma') \quad (5.27)$$

By introducing the fermionic momenta

$$\rho \equiv \frac{\delta \mathcal{L}_2}{\delta \dot{\psi}} = -\frac{ic}{2} (2\mathcal{P}_- + \mathcal{P}_+) \psi^* \quad (5.28)$$

we can write the following anticommutation relation

$$\{\psi_\alpha(\tau, \sigma), \rho_\beta(\tau, \sigma')\} = -2\pi i \alpha' \left(\frac{1+\Gamma_{11}}{2} (\mathcal{P}_- + \mathcal{P}_+) \right)_{\alpha\beta} \delta(\sigma - \sigma') \quad (5.29)$$

The quadratic fermionic Hamiltonian is therefore

$$\mathcal{H}_{2,F} = \frac{i}{4c^2} (c^2 \psi_+ \psi'_+ - 4\rho_+ \rho'_+ + 2c^2 \psi_- \psi'_- - 2\rho_- \rho'_-) - \frac{i}{2} \psi_+ \rho_+ + i\psi_- \rho_- + \frac{1}{2} \psi_- \Gamma_{56} \rho_- \quad (5.30)$$

where we have defined $\rho_\pm = \mathcal{P}_\pm \rho$. The fermionic spectrum can then be computed and reads

$$cH_{2,F} = \sum_{n \in \mathbb{Z}} \left[\sum_{b=1}^4 \omega_n F_n^{(b)} + \sum_{b=5}^6 \left(\Omega_n + \frac{c}{2} \right) F_n^{(b)} + \sum_{b=7}^8 \left(\Omega_n - \frac{c}{2} \right) F_n^{(b)} \right] \quad (5.31)$$

with the number operators $F_n^{(b)} = d_{n,\alpha}^\dagger d_{n,\alpha}$ for $b = 1, \dots, 4$, and $F_n^{(b)} = b_{n,\alpha}^\dagger b_{n,\alpha}$ for $b = 5, \dots, 8$.

The level-matching condition, including also the bosonic part, is

$$\sum_{n \in \mathbb{Z}} n \left[\sum_{i=1}^4 \hat{N}_n^i + \sum_{a=1}^2 (M_n^a + N_n^a) + \sum_{b=1}^8 F_n^{(b)} \right] = 0 \quad (5.32)$$

As mentioned before, the fermionic spectrum splits into four light and four heavy modes. The pp-wave spectrum (5.31) is in agreement with the one computed in [27, 28, 29] in a different coordinate system. However the result of [27, 28, 29] has been obtained using the κ -symmetry gauge fixing condition $\Gamma^+ \theta = 0$. This differs from the gauge choice (4.24) used in this paper which is the appropriate one if one wants to study corrections to the pp-wave limit. In fact the gauge condition (4.24) ensures that we are selecting the appropriate supersymmetric states.

6 Fermionic field redefinition on the Lagrangian

The Lagrangian (4.6)-(4.8), (4.28)-(4.31) found in Section 4 is the full Lagrangian to order $1/R^2$. However, it is convenient to perform fermionic field redefinitions on this Lagrangian to make it easier to pass to a Hamiltonian formalism. In particular, the main complications for the fermionic terms in passing to the Hamiltonian formalism are to change the fermionic variables to fermionic positions and momenta and to perform the Dirac procedure. For both these procedures the relevant quantities to consider are the fermionic momenta. However, it is important to notice that it is the fermionic momenta as functions of the bosonic positions and momenta, as opposed to the bosonic positions and velocities, which are the relevant quantities for both the change of fermionic variables and the Dirac procedure.

The goal of performing fermionic field redefinitions of the Lagrangian is thus that the fermionic momenta, with the bosonic variables being the bosonic positions and momenta, are as simple as possible in terms of their $1/R$ and $1/R^2$ corrections. Unfortunately, it does not seem possible in a straightforward manner to remove $1/R$ corrections to the fermionic momenta by fermionic field redefinitions. This is because the field redefinitions induce problematic terms with second derivatives of the bosonic coordinates at order $1/R^2$. Therefore our goal in the following is to perform field redefinitions of the Lagrangian such that the fermionic momenta have no $1/R^2$ corrections.

Given the Lagrangian $\mathcal{L}_{\text{gf}}(X^i, \dot{X}^i, X'^i, \theta, \dot{\theta}, \theta')$ consider now a field redefinition $\tilde{\theta} = \tilde{\theta}(X^i, \dot{X}^i, X'^i, \theta)$ such that the new Lagrangian is given by

$$\mathcal{L}_{\text{new}}(X^i, \dot{X}^i, X'^i, \theta, \dot{\theta}, \theta') = \mathcal{L}_{\text{gf}}(X^i, \dot{X}^i, X'^i, \tilde{\theta}, \dot{\tilde{\theta}}, \tilde{\theta}') \quad (6.1)$$

Take $\tilde{\theta} = \tilde{\theta}(X^i, \dot{X}^i, X'^i, \theta)$ to be of the form

$$\tilde{\theta} = \theta + \frac{1}{R^2} K^a (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_a \theta + \frac{1}{R^2} \tilde{K}^a (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_{11} \Gamma_a \theta \quad (6.2)$$

where a is summing over $+, -, 1, 2, \dots, 8$ and $K^a = K^a(X^i, \dot{X}^i, X'^i, \theta)$, $\tilde{K}^a = \tilde{K}^a(X^i, \dot{X}^i, X'^i, \theta)$ do not carry any spinor indices. Then we see that

$$\tilde{\theta}(\mathcal{P}_+ + 2\mathcal{P}_-) \dot{\tilde{\theta}} = A_{+, \tau} + \frac{2}{R^2} (K^a A_{a, \tau} + \tilde{K}^a \tilde{A}_{a, \tau}) + \mathcal{O}(R^{-3}) \quad (6.3)$$

Notice that the terms with \dot{K}^a and $\dot{\tilde{K}}^a$ vanish since $\tilde{\theta} \Gamma_a \theta = 0$ and $\tilde{\theta} \Gamma_{11} \Gamma_a \theta = 0$. Similarly we have

$$\tilde{\theta} \Gamma^0 \Gamma_{11} \Gamma_+ \tilde{\theta}' = \tilde{A}_{+, \sigma} + \frac{2}{R^2} (K^a \tilde{A}_{a, \sigma} + \tilde{K}^a A_{a, \sigma}) + \mathcal{O}(R^{-3}) \quad (6.4)$$

$$\tilde{\theta} \Gamma^0 \Gamma_+ \Gamma_{56} \tilde{\theta} = B_{+56} + \frac{2}{R^2} (K^a B_{a56} + \tilde{K}^a \tilde{B}_{a56}) + \mathcal{O}(R^{-3}) \quad (6.5)$$

$$\tilde{\theta} \Gamma_{123} (\mathcal{P}_+ + 4\mathcal{P}_-) \tilde{\theta} = C_{++} + \frac{2}{R^2} (K^a C_{+a} - \tilde{K}^a \tilde{C}_{+a}) + \mathcal{O}(R^{-3}) \quad (6.6)$$

where we used Eqs. (5.13)-(5.15). Thus, the field redefinition (6.2) induces the following additional terms that we should add to the original Lagrangian

$$\begin{aligned} & \frac{ic}{R^2} \left[K^a (A_{a,\tau} + \tilde{A}_{a,\sigma}) + \tilde{K}^a (\tilde{A}_{a,\tau} + A_{a,\sigma}) \right] + \frac{ic^2}{4R^2} \left[K^a (B_{a56} + B_{a78}) + \tilde{K}^a (\tilde{B}_{a56} + \tilde{B}_{a78}) \right] \\ & + \frac{ic^2}{2R^2} (-K^a C_{+a} + \tilde{K}^a \tilde{C}_{+a}) \end{aligned} \quad (6.7)$$

We choose now the following field redefinition

$$j = 1, 2, 3, 4 : \quad K^j = -\frac{i}{2} C_{+j} + \frac{i}{2c} \tilde{A}_{j,\sigma}, \quad \tilde{K}^j = \frac{i}{2c} A_{j,\sigma} \quad (6.8)$$

$$\begin{aligned} j = 5, 6, 7, 8 : \quad K^j &= -\frac{1}{c} s_j u_4 \dot{X}^j - \frac{i}{2} C_{+j} + \frac{i}{2c} \tilde{A}_{j,\sigma} - \frac{i}{8} \sum_{k=5}^8 \epsilon_{jk} B_{k+-} + \frac{i}{4} s_j B_{4j+} \\ \tilde{K}^j &= -\frac{1}{c} s_j u_4 X'^j + \frac{i}{2c} A_{j,\sigma} \end{aligned} \quad (6.9)$$

$$\begin{aligned} K^+ &= -\frac{1}{2c^2} \sum_{i=1}^8 \left[(\dot{X}^i)^2 + (X'^i)^2 \right] - u_4^2 + \frac{i}{2c} \tilde{A}_{+,\sigma} - \frac{i}{4c} \tilde{A}_{-,\sigma} + \frac{i}{4} (B_{+56} + B_{+78} - C_{++} + C_{+-}) \\ \tilde{K}^+ &= -\frac{1}{c^2} \sum_{i=1}^8 \dot{X}^i X'^i - \frac{i}{2c} A_{+,\sigma} - \frac{i}{4c} A_{-,\sigma} \\ K^- &= -\frac{1}{2c^2} \sum_{i=1}^8 \left[(\dot{X}^i)^2 + (X'^i)^2 \right] + \frac{i}{4c} \tilde{A}_{+,\sigma} \\ \tilde{K}^- &= \frac{1}{c^2} \sum_{i=1}^8 \dot{X}^i X'^i + \frac{i}{4c} A_{+,\sigma} \end{aligned} \quad (6.10)$$

This field redefinition gets rid of all $1/R^2$ terms in the fermionic momenta when written in terms of the bosonic positions and momenta, apart from terms coming from the $\bar{\theta} \Gamma_+ \mathcal{M}^2 \dot{\theta}$ kinetic term in (4.31). To remove these terms we perform the additional field redefinition

$$\tilde{\theta} = \theta - \frac{1}{12R^2} \left[\bar{\theta} \Gamma_+ \mathcal{M}^2 (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \right]^T \quad (6.11)$$

This induces the following additional terms that we should add to the four-fermion Lagrangian (4.31)

$$-\frac{ic}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 \dot{\theta} + \frac{ic}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 \Gamma_{11} \theta' - \frac{ic^2}{48} \bar{\theta} \Gamma_+ \mathcal{M}^2 (\Gamma_{56} + \Gamma_{78}) \theta + \frac{ic^2}{24} \bar{\theta} \Gamma_+ \mathcal{M}^2 P \Gamma_{0123} \Gamma_+ \theta \quad (6.12)$$

We now get the following modified Lagrangian after the field redefinition

$$\begin{aligned}
\mathcal{L}_{4,BF} = & -\frac{i}{4} \sum_{i=1}^8 \left[(\dot{X}^i)^2 + (X'^i)^2 \right] \left(\frac{4}{c} \tilde{A}_{+,\sigma} + B_{+56} + B_{+78} - C_{++} + C_{+-} \right) - \frac{ic^2}{2} \sum_{i=1}^3 u_i^2 C_{++} \\
& + ic \tilde{A}_{+,\sigma} \left[\sum_{i=1}^3 u_i^2 - u_4^2 \right] + \frac{ic^2}{4} \sum_{i=1}^4 u_i^2 (B_{+56} + B_{+78}) - 2iu_4 \sum_{i=5}^8 s_i (A_{i,\tau} + \tilde{A}_{i,\sigma}) \dot{X}^i \\
& - \frac{i}{2} \sum_{i=1}^8 \dot{X}^i X'^i \left(\frac{4}{c} A_{+,\sigma} + \tilde{B}_{+56} + \tilde{B}_{+78} + \tilde{C}_{+-} \right) + i \sum_{i,j=1}^8 \left[C_{ij} (X'^i X'^j - \dot{X}^i \dot{X}^j) + 2\tilde{C}_{ij} X'^i \dot{X}^j \right] \\
& + \frac{ic}{2} u_4 \sum_{i=5}^8 s_i (3C_{+i} \dot{X}^i + \tilde{C}_{+i} X'^i) + \frac{ic}{2} u_4 \sum_{i=1}^8 \left[(B_{i56} - B_{i78}) \dot{X}^i - (\tilde{B}_{i56} - \tilde{B}_{i78}) X'^i \right] \\
& - \frac{ic}{2} \sum_{i=1}^8 \sum_{j=1}^4 u_j (B_{-ij} \dot{X}^i - \tilde{B}_{-ij} X'^i) - \frac{i}{2} \sum_{i=1}^8 \sum_{j=5}^8 s_j \left[(\dot{X}^i \dot{X}^j - X'^i X'^j) B_{4ij} + (\dot{X}^i X'^j - X'^i \dot{X}^j) \tilde{B}_{4ij} \right] \\
& + \frac{ic}{2} \sum_{i=1}^3 \sum_{j=4}^8 u_i \left[B_{+ij} \dot{X}^j - \tilde{B}_{+ij} X'^j \right] - \frac{ic}{4} (B_{+56} \dot{x}_1 y_1 + \tilde{B}_{+56} x'_1 y_1 + B_{+78} \dot{x}_2 y_2 + \tilde{B}_{+78} x'_2 y_2) \\
& + \frac{i}{4} \sum_{i=1}^8 \sum_{j,k=5}^8 \epsilon_{jk} \left[(B_{+ij} - B_{-ij}) (\dot{X}^i \dot{X}^k - X'^i X'^k) + (\tilde{B}_{+ij} - \tilde{B}_{-ij}) (\dot{X}^i X'^k - X'^i \dot{X}^k) \right] \\
& + \frac{ic}{4} u_4 \sum_{i=5}^8 (-B_{+4i} \dot{X}^i + 3\tilde{B}_{+4i} X'^i) - \frac{ic}{8} u_4 \sum_{i,j=5}^8 s_i \epsilon_{ij} (B_{+i-} \dot{X}^j + \tilde{B}_{+i-} X'^j) \\
& + \frac{ic}{2} u_4 \sum_{i=1}^3 (B_{+i4} \dot{u}_i - \tilde{B}_{+i4} u'_i) + ic \sum_{i,j=1}^3 u'_i u_j \tilde{B}_{+ij} - \frac{ic}{2} u_4 \sum_{i=5}^8 s_i (B_{i56} + B_{i78}) \dot{X}^i
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
\mathcal{L}_{4,F} = & \frac{ic}{12} \bar{\theta} \Gamma_{11} \Gamma_+ \mathcal{M}^2 \theta' + \frac{ic}{12} \bar{\theta} \Gamma_+ \mathcal{M}^2 \Gamma_{11} \theta' - \frac{1}{2} \sum_{i=1}^8 \left[A_{i,\tau} + \tilde{A}_{i,\sigma} + \frac{c}{4} (B_{i56} + B_{i78}) - c C_{+i} \right]^2 \\
& + \frac{c^2}{8} \sum_{i=1}^8 C_{+i}^2 - \frac{c}{4} \sum_{i=5}^8 \left[s_i B_{+4i} - \frac{1}{2} \sum_{j=5}^8 \epsilon_{ij} B_{+-j} \right] \left(A_{i,\tau} + \tilde{A}_{i,\sigma} - \frac{c}{2} C_{+i} + \frac{c}{4} (B_{i56} + B_{i78}) \right) \\
& + \frac{1}{2} (A_{+,\sigma}^2 - \tilde{A}_{+,\sigma}^2) + \frac{c}{4} A_{+,\sigma} \left[\tilde{C}_{+-} + \tilde{B}_{+56} + \tilde{B}_{+78} \right] - \frac{c}{4} \tilde{A}_{+,\sigma} \left[C_{+-} - C_{++} + B_{+56} + B_{+78} \right]
\end{aligned} \tag{6.14}$$

In terms of this new Lagrangian we have

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_{j,\tau}} &= \frac{i}{R} p_j, \quad \frac{\partial \mathcal{L}}{\partial \tilde{A}_{j,\tau}} = -\frac{i}{R} X'^j \\
\frac{\partial \mathcal{L}}{\partial A_{+,\tau}} &= \frac{ic}{2}, \quad \frac{\partial \mathcal{L}}{\partial \tilde{A}_{+,\tau}} = \frac{\partial \mathcal{L}}{\partial A_{-,\tau}} = \frac{\partial \mathcal{L}}{\partial \tilde{A}_{-,\tau}} = 0 \\
\frac{\partial \mathcal{L}}{\partial (\bar{\theta} \Gamma_+ \mathcal{M}^2 \theta)} &= 0
\end{aligned} \tag{6.15}$$

with $j = 1, 2, \dots, 8$ and where we used

$$p_{i=1\dots 4} = (p_{u_1}, p_{u_2}, p_{u_3}, p_{u_4}), \quad p_{i=5\dots 8} = \frac{\sqrt{2}}{4} (p_{x_1}, p_{y_1}, p_{x_2}, p_{y_2}) \tag{6.16}$$

From (6.15) we see that the purpose of making the fermionic field redefinitions is fulfilled in that the fermionic momenta do not have $1/R^2$ corrections and that they do not depend on $\dot{\theta}$. This ensures that one needs only to take into account the $1/R$ corrections to the fermionic momenta when changing variables to fermionic phase space variables in the Hamiltonian and when performing the Dirac procedure. Furthermore any $\dot{\theta}$ dependence in the fermionic momenta would have caused problems in these procedures.

7 Full light cone Hamiltonian up to quartic terms

This section contains the two principle results of this paper. The first is a way to combine the shift to fermionic phase space variables and the Dirac procedure (getting the Hamiltonian in a form where the canonical quantization is simple). This is a highly non-trivial result since we have both a first and a second order correction to the Hamiltonian.

The second principle result is that we have computed explicitly the full Hamiltonian, including quadratic, cubic and quartic terms in the fields, in phase space variables with a simple canonical quantization procedure. The total Hamiltonian that we compute in this paper is written in the following way

$$\mathcal{H} = \mathcal{H}_{2,B} + \mathcal{H}_{2,F} + \frac{1}{R}(\mathcal{H}_{3,B} + \mathcal{H}_{3,BF}) + \frac{1}{R^2}(\mathcal{H}_{4,B} + \mathcal{H}_{4,BF} + \mathcal{H}_{4,F}) + \mathcal{O}(R^{-3}) \quad (7.1)$$

7.1 Cubic and quartic terms in bosonic Hamiltonian

The bosonic terms in the Hamiltonian are readily obtained using the Legendre transform

$$c\mathcal{H}_B = \sum_{i=1}^4 \dot{u}_i \Pi_{u_i} + \sum_{i=1}^2 \dot{x}_i \Pi_{x_i} + \sum_{i=1}^2 \dot{y}_i \Pi_{y_i} - \mathcal{L}_B \quad (7.2)$$

where \mathcal{L}_B is the bosonic part of the light cone gauge fixed Lagrangian (4.6)-(4.8) and where

$$\Pi_\mu = \frac{\partial \mathcal{L}_B}{\partial \dot{x}^\mu} \quad (7.3)$$

For convenience we define as in Section 5

$$p_{x_a} \equiv 8\Pi_{x_a} - cy_a, \quad p_{y_a} \equiv 8\Pi_{y_a}, \quad p_{u_i} \equiv \Pi_{u_i} \quad (7.4)$$

Notice that these fields are functions of the momenta and position variables. The cubic terms in the bosonic Hamiltonian are found as

$$\mathcal{H}_{3,B} = \frac{u_4}{8c} \left[p_{x_1}^2 + p_{y_1}^2 - p_{x_2}^2 - p_{y_2}^2 - x_1'^2 - y_1'^2 + x_2'^2 + y_2'^2 \right] \quad (7.5)$$

The quartic terms in the bosonic Hamiltonian are

$$\begin{aligned} \mathcal{H}_{4,B} = & \frac{2}{c^3} \left(\sum_{i=1}^8 p_i X'^i \right)^2 - \frac{1}{2c^3} \left(\sum_{i=1}^8 (p_i^2 + (X'^i)^2) - c^2 \sum_{i=1}^3 u_i^2 + c^2 u_4^2 \right)^2 \\ & + c \left(\sum_{i=1}^3 u_i^2 \right)^2 + \frac{4}{3} c u_4^4 + \frac{1}{c} \sum_{i,j=1}^3 u_i^2 (u_j'^2 - p_j^2) + \frac{2}{c} u_4^2 \sum_{i=5}^8 p_i^2 \end{aligned}$$

$$+\frac{1}{12\sqrt{2}}(p_5 y_1^3 + p_7 y_2^3) + \frac{1}{2c} y_1^2(p_5^2 - X_5'^2) + \frac{1}{2c} y_2^2(p_7^2 - X_7'^2) \quad (7.6)$$

with

$$\begin{aligned} p_{i=1\dots 4} &= (p_{u_1}, p_{u_2}, p_{u_3}, p_{u_4}) , \quad p_{i=5\dots 8} = \frac{\sqrt{2}}{4}(p_{x_1}, p_{y_1}, p_{x_2}, p_{y_2}) \\ X'^{i=1\dots 4} &= (u'_1, u'_2, u'_3, u'_4) , \quad X'^{i=5\dots 8} = \frac{\sqrt{2}}{4}(x'_1, y'_1, x'_2, y'_2) \end{aligned} \quad (7.7)$$

Thus, (5.4) along with (7.5)-(7.6) constitute the final expression for the bosonic part of the Hamiltonian.

7.2 Preliminary expressions for the fermionic Hamiltonian

In this section we compute the fermionic part of the Hamiltonian as function of the bosonic phase space variables (*i.e.* position and momenta) but without changing variables to fermionic phase space variables. This change of variables is performed below in Section 7.3 where we also perform the Dirac procedure. We find the Hamiltonian using the field redefined Lagrangian given by (5.18), (4.29) and (6.13)-(6.14).

Before finding the Hamiltonian we perform first the variable shift in the Lagrangian from θ to ψ and ψ^* defined by

$$\psi = \theta^1 + i\Gamma_{049}\theta^2 , \quad \psi^* = \theta^1 - i\Gamma_{049}\theta^2 \quad (7.8)$$

as in Eq. (5.16) in Section 5. In terms of the Lagrangian in these variables we define the fermionic momenta

$$\rho = \frac{\partial \mathcal{L}_{\text{gf}}}{\partial \dot{\psi}} , \quad \rho^* = \frac{\partial \mathcal{L}_{\text{gf}}}{\partial \dot{\psi}^*} \quad (7.9)$$

We find ρ and ρ^* explicitly in Section 7.3. The Hamiltonian is now found by employing the Legendre transform

$$\mathcal{H} = \dot{\psi}\rho + \dot{\psi}^*\rho^* + \sum_{i=1}^4 \dot{u}_i \Pi_{u_i} + \sum_{i=1}^2 \dot{x}_i \Pi_{x_i} + \sum_{i=1}^2 \dot{y}_i \Pi_{y_i} - \mathcal{L}_{\text{gf}} \quad (7.10)$$

We can perform the Legendre transform without knowing explicitly ρ and ρ^* in terms of ψ and ψ^* . All we need to employ is that we know from (6.15) that when the Lagrangian is written in bosonic phase space variables there are no terms that are non-linear in $\dot{\theta}$. Hence the fermionic part in the Legendre transform (7.10) simply corresponds to removing all terms with $\dot{\theta}$ in the Lagrangian written in bosonic phase space variables.

From the above we compute the cubic fermionic part of the Hamiltonian to be⁴

$$\begin{aligned} \mathcal{H}_{3,BF} &= -\frac{ic}{4}B_{+-4}u_4 - \frac{ic}{4}(B_{+56} - B_{+78})u_4 - \frac{i}{4}\sum_{i=5}^8 s_i(B_{+4i}p_i + \tilde{B}_{+4i}X'^i) \\ &+ i\sum_{i=1}^8 \left[\left(-\frac{1}{c}\tilde{A}_{i,\sigma} + C_{+i} - \frac{1}{4}(B_{i56} + B_{i78}) \right) p_i + \left(\frac{1}{c}A_{i,\sigma} + \tilde{C}_{+i} + \frac{1}{4}(\tilde{B}_{i56} + \tilde{B}_{i78}) \right) X'^i \right] \\ &- \frac{i}{8}\sum_{i,j=5}^8 \epsilon_{ij}(B_{+-i}p_j + \tilde{B}_{+-i}X'^j) \end{aligned} \quad (7.11)$$

⁴In deriving this we used that $B_{+-i} = 0$ for $i = 1, 2, 3$.

The part with two bosons and two fermions is

$$\begin{aligned}
\mathcal{H}_{4,BF} = & \frac{i}{c^2} \sum_{i=1}^8 \left(p_i^2 + (X'^i)^2 \right) \left[\tilde{A}_{+,\sigma} + \frac{c}{4} (B_{+56} + B_{+78} - C_{++} + C_{+-}) \right] - i \tilde{A}_{+,\sigma} \left[\sum_{i=1}^3 u_i^2 - u_4^2 \right] \\
& + \frac{2i}{c^2} \sum_{i=1}^8 p_i X'^i \left[A_{+,\sigma} + \frac{c}{4} (\tilde{B}_{+56} + \tilde{B}_{+78}) + \frac{c}{4} \tilde{C}_{+-} \right] + \frac{ic}{2} \sum_{i=1}^3 u_i^2 C_{++} - \frac{ic}{4} \sum_{i=1}^4 u_i^2 (B_{+56} + B_{+78}) \\
& + \frac{i}{2} u_4 \sum_{i=5}^8 s_i \left[C_{+i} p_i - \tilde{C}_{+i} X'^i \right] - \frac{i}{c} \sum_{i,j=1}^8 \left[C_{ij} (X'^i X'^j - p_i p_j) + 2 \tilde{C}_{ij} X'^i p_j \right] - i \sum_{i,j=1}^3 u'_i u_j \tilde{B}_{+ij} \\
& - \frac{i}{2} u_4 \sum_{i=1}^8 \left[(B_{i56} - B_{i78}) p_i - (\tilde{B}_{i56} - \tilde{B}_{i78}) X'^i \right] - \frac{i}{8} u_4 \sum_{i,j=5}^8 s_i \epsilon_{ij} (3 B_{+-i} p_j + \tilde{B}_{+-i} X'^j) \\
& - \frac{i}{2} \sum_{i=1}^4 \sum_{j=1}^8 u_i \left[B_{-ij} p_j - \tilde{B}_{-ij} X'^j \right] + \frac{i}{2c} \sum_{i=1}^8 \sum_{j=5}^8 s_j \left[(p_i p_j - X'^i X'^j) B_{4ij} + (p_i X'^j - X'^i p_j) \tilde{B}_{4ij} \right] \\
& - \frac{i}{2} \sum_{i=1}^3 \sum_{j=4}^8 u_i \left[B_{+ij} p_j - \tilde{B}_{+ij} X'^j \right] + \frac{i}{4} (B_{+56} p_{x_1} y_1 + \tilde{B}_{+56} x'_1 y_1 + B_{+78} p_{x_2} y_2 + \tilde{B}_{+78} x'_2 y_2) \\
& - \frac{i}{4c} \sum_{i=1}^8 \sum_{j,k=5}^8 \epsilon_{jk} \left[(B_{+ij} - B_{-ij}) (p_i p_k - X'^i X'^k) + (\tilde{B}_{+ij} - \tilde{B}_{-ij}) (p_i X'^k - X'^i p_k) \right] \\
& - \frac{i}{4} u_4 \sum_{i=5}^8 (B_{+4i} p_i + 3 \tilde{B}_{+4i} X'^i) + \frac{i}{2} u_4 \sum_{i=1}^3 (B_{+4i} p_i - \tilde{B}_{+4i} u'_i)
\end{aligned} \tag{7.12}$$

Finally, the four-fermion part is

$$\begin{aligned}
\mathcal{H}_{4,F} = & -\frac{i}{12} \left(\bar{\theta} \Gamma_{11} \Gamma_+ \mathcal{M}^2 \theta' + \bar{\theta} \Gamma_+ \mathcal{M}^2 \Gamma_{11} \theta' \right) - \frac{1}{2c} (A_{+,\sigma}^2 - \tilde{A}_{+,\sigma}^2) \\
& - \frac{1}{4} A_{+,\sigma} (\tilde{C}_{+-} + \tilde{B}_{+56} + \tilde{B}_{+78}) + \frac{1}{4} \tilde{A}_{+,\sigma} (C_{+-} - C_{++} + B_{+56} + B_{+78}) \\
& - \frac{c}{8} \sum_{i=1}^4 C_{+i}^2 - \frac{c}{32} \sum_{i=5}^8 \left[2 C_{+i} - s_i B_{+4i} + \frac{1}{2} \sum_{j=5}^8 \epsilon_{ij} B_{+-j} \right]^2
\end{aligned} \tag{7.13}$$

Note that all the above expressions (7.11)-(7.13) can be thought of as being in terms of ψ and ψ^* rather than in terms of θ . More explicitly, one can invert (7.8)

$$\theta = \theta^1 + \theta^2, \quad \theta^1 = \frac{1}{2}(\psi + \psi^*), \quad \theta^2 = \frac{\Gamma_{049}}{2i}(\psi - \psi^*) \tag{7.14}$$

and plug this into (4.10) in order to obtain $A_{a,\sigma}$, $\tilde{A}_{a,\sigma}$, B_{abc} , \tilde{B}_{abc} , C_{ab} and \tilde{C}_{ab} in terms of ψ and ψ^* .

7.3 Dirac procedure and shift to fermionic phase space variables

As stated in Section 7.2 the Hamiltonian (7.11)-(7.13) is not our final expression for the fermionic terms in the Hamiltonian. To find the final form we need to perform two tasks, both involving a fermionic field redefinition of the Hamiltonian. First we should make the coordinate change from (ψ, ψ^*) to the fermionic phase space variables (ψ, ρ) . However, this is not sufficient since the canonical commutation relations would have a complicated structure

due to the fact that the Dirac brackets receive $1/R$ and $1/R^2$ corrections. Hence the second task is to determine the field redefinition that one should perform on the Hamiltonian in order to be able to use the canonical commutation relations without $1/R$ and $1/R^2$ corrections.

Performing these two tasks, and finding the two fermionic field redefinitions, is very non-trivial and involved since we have both $1/R$ and $1/R^2$ corrections. We found therefore a way to combine the two field redefinitions into one field redefinition performed on the θ variable. In this way the structure of the combined field redefinition becomes sufficiently elegant and simple such that we can perform it explicitly and compute the resulting extra terms that one should add to the Hamiltonian (7.11)-(7.13).

Computing the fermionic momenta

To proceed with the program outlined above we need to compute the fermionic momenta as defined in (7.9). From the field redefined Lagrangian (5.18), (4.29) and (6.13)-(6.14) we see that the terms with $\dot{\theta}$ are

$$\mathcal{L}_{\text{gf}} = -\frac{ic}{2}\dot{\psi}(\mathcal{P}_+ + 2\mathcal{P}_-)\psi^* + \frac{i}{R}\sum_{i=1}^8(p_i A_{i,\tau} - X'^i \tilde{A}_{i,\tau}) + \mathcal{O}(R^{-3}) \quad (7.15)$$

From this we find the fermionic momenta to be of the form

$$\rho = E(\psi^* + m\psi^* + n\psi) , \quad \rho^* = E(m\psi + n\psi^*) \quad (7.16)$$

with E , m and n determined as

$$\begin{aligned} E &= -\frac{ic}{2}(\mathcal{P}_+ + 2\mathcal{P}_-) \\ m + n &= \frac{1}{cR}\sum_{i=1}^8(p_i + X'^i)(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_i(\mathcal{P}_+ + \mathcal{P}_-) \\ \Gamma_{049}(m - n)\Gamma_{049} &= \frac{1}{cR}\sum_{i=1}^8(p_i - X'^i)(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-)\Gamma^0\Gamma_i(\mathcal{P}_+ + \mathcal{P}_-) \end{aligned} \quad (7.17)$$

Shift to fermionic phase space variables

Consider the Hamiltonian listed in Eqs. (7.11)-(7.13). This is the fermionic part of the Hamiltonian written in terms of ψ and ψ^* as well as in the bosonic phase space variables. We write this as

$$\mathcal{H}_{(1)}(\psi, \psi^*) \quad (7.18)$$

Our first task is to write this Hamiltonian in the phase space variables ψ and ρ . This one can do by eliminating ψ^* order by order in $1/R$ using the expressions for $\rho(\psi, \psi^*)$ in (7.16)-(7.17). This can equivalently be thought of as a fermionic field redefinition of ψ and ψ^* . Specifically, by inverting (7.16) the field redefinition takes the form

$$\psi(\psi, \rho) = \psi , \quad \psi^*(\psi, \rho) = E^{-1}\rho - mE^{-1}\rho - n\psi + m^2E^{-1}\rho + mn\psi \quad (7.19)$$

Here E^{-1} is defined by $EE^{-1} = E^{-1}E = \mathcal{P}_+ + \mathcal{P}_-$. Then we define the Hamiltonian

$$\mathcal{H}_{(2)}(\psi, \rho) \equiv \mathcal{H}_{(1)}(\psi, \psi^*(\psi, \rho)) \quad (7.20)$$

This is the Hamiltonian written correctly in terms of the phase space variables ψ and ρ .

Dirac procedure

The second task is to implement the Dirac procedure. This is necessary since the fermionic momenta (7.16)-(7.17) results in a complicated structure for the canonical commutation relations since the Dirac brackets are non-trivial.

We begin by defining the Poisson bracket for Grassmanian fields as

$$\{A, B\}_P = -(\mathcal{P}_- + \mathcal{P}_+)_{\alpha\beta} \left[\left(\frac{\partial A}{\partial \psi_\alpha} \frac{\partial B}{\partial \rho_\beta} + \frac{\partial B}{\partial \psi_\alpha} \frac{\partial A}{\partial \rho_\beta} \right) + \left(\frac{\partial A}{\partial \psi_\alpha^\dagger} \frac{\partial B}{\partial \rho_\beta^\dagger} + \frac{\partial B}{\partial \psi_\alpha^\dagger} \frac{\partial A}{\partial \rho_\beta^\dagger} \right) \right] \quad (7.21)$$

Here the projector $\mathcal{P}_+ + \mathcal{P}_-$ is due to the fact that we work in the subspace of physical states. From the fermionic momenta (7.16)-(7.17) we read off the constraints (up to corrections of order R^{-3})

$$\begin{aligned} \eta^1(\psi, \psi^*, \rho, \rho^*) &= \rho - E\psi^* - Em\psi^* - En\psi \\ \eta^2(\psi, \psi^*, \rho, \rho^*) &= \rho^* - Em\psi - En\psi^* \end{aligned} \quad (7.22)$$

In terms of these constraints the Dirac bracket for Grassmanian fields is defined as

$$\{A, B\}_D = \{A, B\}_P - \{A, \eta_\alpha^i\}_P (C^{-1})^{(i\alpha)(j\beta)} \{\eta_\beta^j, B\}_P \quad (7.23)$$

where $C_{(i\alpha)(j\beta)} = \{\eta_\alpha^i, \eta_\beta^j\}_P$. We compute

$$C = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} + 2E \begin{pmatrix} n & m \\ m & n \end{pmatrix} \quad (7.24)$$

The inverse is

$$C^{-1} = \begin{pmatrix} 0 & E^{-1} \\ E^{-1} & 0 \end{pmatrix} - 2 \begin{pmatrix} n & m \\ m & n \end{pmatrix} E^{-1} + 4 \begin{pmatrix} mn + nm & m^2 + n^2 \\ m^2 + n^2 & mn + nm \end{pmatrix} E^{-1} \quad (7.25)$$

up to corrections of order R^{-3} . Using these results we find the Dirac brackets

$$\begin{aligned} \{\psi_\alpha, \psi_\beta\}_D &= [2nE^{-1} - 4(mn + nm)E^{-1}]_{\alpha\beta} \\ \{\psi_\alpha, \rho_\beta\}_D &= \{\psi_\alpha, \rho_\beta\}_P + [m - 2m^2 - 2n^2]_{\alpha\beta} \end{aligned} \quad (7.26)$$

To quantize the theory, one should impose the above Dirac brackets for ψ and ρ as the anti-commutators for the quantum fields. However, one can also find a field redefinition $\tilde{\psi}(\psi, \rho)$ and $\tilde{\rho}(\psi, \rho)$ such that the Poisson brackets for $\tilde{\psi}$ and $\tilde{\rho}$ are equal to the Dirac brackets (7.26), *i.e.*

$$\{\psi_\alpha, \psi_\beta\}_D = \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\}_P, \quad \{\psi_\alpha, \rho_\beta\}_D = \{\tilde{\psi}_\alpha, \tilde{\rho}_\beta\}_P \quad (7.27)$$

We write this field redefinition as

$$\tilde{\psi}(\psi, \rho) = \psi + A\psi + B\rho, \quad \tilde{\rho}(\psi, \rho) = \rho + \tilde{A}\rho + \tilde{B}\psi \quad (7.28)$$

and define the Hamiltonian

$$\mathcal{H}_{(3)}(\psi, \rho) \equiv \mathcal{H}_{(2)}(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho)) \quad (7.29)$$

Note here that $\mathcal{H}_{(2)}(\psi, \rho)$ obviously is the Hamiltonian that one should perform the field redefinition in since this is the one written in terms of the phase space variables. We compute

$$\begin{aligned} \psi^*(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho)) &= E^{-1}\rho + (-mE^{-1} + E^{-1}\tilde{A} - mE^{-1}\tilde{A} - nB + m^2E^{-1})\rho \\ &\quad + (-n + E^{-1}\tilde{B} - nA - mE^{-1}\tilde{B} + mn)\psi \end{aligned} \quad (7.30)$$

Imposing now (7.27) we see that the field redefinition (7.28) is required to satisfy

$$\begin{aligned} \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\}_P &= \{\psi_\alpha, \psi_\beta\}_D = [(2n - 4mn - 4nm)E^{-1}]_{\alpha\beta} \\ \{\tilde{\psi}_\alpha, \tilde{\rho}_\beta\}_P &= \{\psi_\alpha, \rho_\beta\}_D = \{\psi_\alpha, \rho_\beta\}_P + [m - 2m^2 - 2n^2]_{\alpha\beta} \end{aligned} \quad (7.31)$$

We compute

$$\begin{aligned} \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta\}_P &= -[B + B^T + AB^T + BA^T]_{\alpha\beta} \\ \{\tilde{\psi}_\alpha, \tilde{\rho}_\beta\}_P &= \{\psi_\alpha, \rho_\beta\}_P - [A + \tilde{A}^T + A\tilde{A}^T + B\tilde{B}^T]_{\alpha\beta} \end{aligned} \quad (7.32)$$

and hence we obtain explicitly the constraints on the field redefinition (7.28)

$$\begin{aligned} B + B^T + AB^T + BA^T &= (-2n + 4mn + 4nm)E^{-1} \\ A + \tilde{A}^T + A\tilde{A}^T + B\tilde{B}^T &= -m + 2m^2 + 2n^2 \end{aligned} \quad (7.33)$$

Combining change of variables and Dirac procedure into redefinition of θ

We consider now how to solve the constraints (7.33) on the field redefinition (7.28) in order for (7.27) to be satisfied. We do this by seeing the two field redefinitions (7.19) and (7.28) as one combined redefinition from $\mathcal{H}_{(1)}$ to $\mathcal{H}_{(3)}$

$$\mathcal{H}_{(3)}(\psi, \rho) \equiv \mathcal{H}_{(2)}(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho)) = \mathcal{H}_{(1)}(\tilde{\psi}(\psi, \rho), \psi^*(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho))) \quad (7.34)$$

and demand that this field redefinition should take a simple and elegant form when written as a field redefinition of the variable θ .

We begin with imposing that to first order in $1/R$ our transformation looks like

$$\tilde{\psi}(\psi, \rho) = \psi - m\psi - nE^{-1}\rho + \mathcal{O}(R^{-2}), \quad \psi^*(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho)) = E^{-1}\rho - mE^{-1}\rho - n\psi + \mathcal{O}(R^{-2}) \quad (7.35)$$

From (7.30) we see that this means that \tilde{A} and \tilde{B} are zero at order $1/R$. We get furthermore that

$$A = -m + \mathcal{O}(R^{-2}), \quad B = -nE^{-1} + \mathcal{O}(R^{-2}) \quad (7.36)$$

using here $(nE^{-1})^T = nE^{-1}$. Inserting this in (7.33) we get

$$A = -m + 2m^2 + 2n^2 - \tilde{A}^T, \quad BE = -n + \frac{3}{2}(mn + nm) \quad (7.37)$$

using here $nE^{-1}m^T = nmE^{-1}$. We get therefore

$$\tilde{\psi}(\psi, \rho) = \psi - m\psi - nE^{-1}\rho + (2m^2 + 2n^2 - \tilde{A}^T)\psi + \frac{3}{2}(mn + nm)E^{-1}\rho \quad (7.38)$$

$$\begin{aligned}\psi^*(\tilde{\psi}(\psi, \rho), \tilde{\rho}(\psi, \rho)) &= E^{-1}\rho - mE^{-1}\rho - n\psi + (m^2 + n^2 + E^{-1}\tilde{A}E)E^{-1}\rho \\ &\quad + (nm + mn + E^{-1}\tilde{B})\psi\end{aligned}\quad (7.39)$$

We write now the combined field redefinition of the Hamiltonian (7.34) as

$$\mathcal{H}_{(3)}(\psi, E\psi^*) = \mathcal{H}_{(1)}(\psi + \Delta\psi(\psi, \psi^*), \psi^* + \Delta\psi^*(\psi, \psi^*)) \quad (7.40)$$

Then

$$\begin{aligned}\Delta\psi(\psi, \psi^*) &= -m\psi - n\psi^* + (2m^2 + 2n^2 - \tilde{A})\psi + \frac{3}{2}(mn + nm)\psi^* \\ \Delta\psi^*(\psi, \psi^*) &= -m\psi^* - n\psi + (m^2 + n^2 + E^{-1}\tilde{A}E)\psi^* + (nm + mn + E^{-1}\tilde{B})\psi\end{aligned}\quad (7.41)$$

We would like that $\Delta\psi^*(\psi, \psi^*) = \Delta\psi(\psi^*, \psi)$ since this seems a necessary requirement for writing the field redefinition in a simple fashion in terms of θ . This fixes

$$\tilde{A}^T = E^{-1}\tilde{A}E = \frac{1}{2}(m^2 + n^2), \quad E^{-1}\tilde{B} = \frac{1}{2}(mn + nm) \quad (7.42)$$

Note that this works since $E^{-1}(m^2)^TE = m^2$ and $E^{-1}(n^2)^TE = n^2$. Then the total field redefinition is

$$\begin{aligned}\Delta\psi(\psi, \psi^*) &= -m\psi - n\psi^* + \frac{3}{2}(m^2 + n^2)\psi + \frac{3}{2}(mn + nm)\psi^* \\ \Delta\psi^*(\psi, \psi^*) &= -m\psi^* - n\psi + \frac{3}{2}(m^2 + n^2)\psi^* + \frac{3}{2}(mn + nm)\psi\end{aligned}\quad (7.43)$$

We now wish to write down the equivalent field redefinition in terms of θ . Thus, we wish to perform the field redefinition

$$\mathcal{H}_{(3)}(\theta) = \mathcal{H}_{(1)}(\theta + \Delta\theta(\theta)) \quad (7.44)$$

Using (7.14) we find the field redefinition (7.43) in terms of θ^1 and θ^2

$$\begin{aligned}\Delta\theta^1 &= \left(-(m+n) + \frac{3}{2}(m+n)^2 \right) \theta^1 \\ \Delta\theta^2 &= \left(-\Gamma_{049}(m-n)\Gamma_{049} + \frac{3}{2}(\Gamma_{049}(m-n)\Gamma_{049})^2 \right) \theta^2\end{aligned}\quad (7.45)$$

Finally, from (7.17) we see that we can write the field redefinition for θ on the form

$$\begin{aligned}\Delta\theta &= \left[-(Y + \Gamma_{11}\tilde{Y}) + \frac{3}{2}(Y + \Gamma_{11}\tilde{Y})^2 \right] \theta \\ Y &= \frac{1}{cR} \sum_{i=1}^8 p_i(\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-) \Gamma^0 \Gamma_i (\mathcal{P}_+ + \mathcal{P}_-), \quad \tilde{Y} = \frac{1}{cR} \sum_{i=1}^8 X'^i (\mathcal{P}_+ + \frac{1}{2}\mathcal{P}_-) \Gamma^0 \Gamma_i (\mathcal{P}_+ + \mathcal{P}_-)\end{aligned}\quad (7.46)$$

Extra terms induced by the field redefinition

We now consider what extra terms should be added to the $1/R$ and $1/R^2$ terms of the Hamiltonian (7.11)-(7.13) in accordance with the field redefinition (7.46). Using $\Delta\theta$ with \mathcal{H}_2 we see that we should add the following cubic terms

$$\Delta\mathcal{H}_{3,BF} = \frac{i}{c} \sum_{i=1}^8 (-X'^i A_{i,\sigma} + p_i \tilde{A}_{i,\sigma}) + \frac{i}{4} \sum_{i=1}^8 (p_i B_{i56} - X'^i \tilde{B}_{i56}) - \frac{i}{2} \sum_{i=1}^8 (p_i C_{+i} + X'^i \tilde{C}_{+i}) \quad (7.47)$$

For the quartic terms we have three different sources of contributions. Either from $\Delta\theta$ in \mathcal{H}_2 , from $\Delta\theta$ twice in \mathcal{H}_2 or from $\Delta\theta$ in \mathcal{H}_3 . This gives the following quartic terms that all should be added to $\mathcal{H}_{4,BF}$

$$\begin{aligned}
\Delta\mathcal{H}_{4,BF} = & \frac{i}{2c^2} \sum_{i,j=1}^8 (p_i p'_j + X'^i X''^j) \tilde{E}_{ij} - \frac{i}{2c^2} \sum_{i,j=1}^8 (X'^i p'_j + p_i X''^j) E_{ij} \\
& - \frac{3i}{4c} \sum_{i,j=1}^8 (p_i p_j - X'^i X'^j) \mathcal{C}_{i+;j} + \frac{3i}{4c} \sum_{i,j=1}^8 (X'^i p_j - p_i X'^j) \tilde{\mathcal{C}}_{i+;j} \\
& - \frac{i}{4c} \sum_{i,j=1}^8 (p_i p_j + X'^i X'^j) \mathcal{C}_{+i;j} - \frac{i}{4c} \sum_{i,j=1}^8 (X'^i p_j + p_i X'^j) \tilde{\mathcal{C}}_{+i;j} \\
& + \frac{i}{4c} \sum_{i,j=5}^8 \sum_{k=1}^8 \epsilon_{ij} \left[(p_i p_k + X'^i X'^k) (\mathcal{B}_{+-i;k} + E_{jk}) + (X'^i p_k + p_i X'^k) (\tilde{\mathcal{B}}_{+-i;k} - \tilde{E}_{jk}) \right] \\
& + \frac{i u_4}{2} \sum_{i=1}^8 (p_j \mathcal{B}_{+-4;i} - X'^j \tilde{\mathcal{B}}_{+-4;i}) + \frac{i u_4}{2} \sum_{i=1}^8 \left[p_i (B_{i56} - B_{i78}) - X'^i (\tilde{B}_{i56} - \tilde{B}_{i78}) \right] \\
& + \frac{i}{2c} \sum_{i=5}^8 \sum_{j=1}^8 s_i \left[(p_i p_j - X'^i X'^j) \mathcal{B}_{+4i;j} + (X'^i p_j - p_i X'^j) \tilde{\mathcal{B}}_{+4i;j} \right]
\end{aligned} \tag{7.48}$$

We defined here the purely fermionic terms

$$\begin{aligned}
\mathcal{B}_{abc;d} &= \bar{\theta} \Gamma_{abc} (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_d \theta, \quad \tilde{\mathcal{B}}_{abc;d} = \bar{\theta} \Gamma_{11} \Gamma_{abc} (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_d \theta \\
\mathcal{C}_{ab;c} &= \bar{\theta} \Gamma_a P \Gamma_{0123} \Gamma_b (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_c \theta, \quad \tilde{\mathcal{C}}_{ab;c} = \bar{\theta} \Gamma_{11} \Gamma_a P \Gamma_{0123} \Gamma_b (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_c \theta \\
E_{ab} &= \bar{\theta} \Gamma_a (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_b \theta, \quad \tilde{E}_{ab} = \bar{\theta} \Gamma_{11} \Gamma_a (\mathcal{P}_+ + \frac{1}{2} \mathcal{P}_-) \Gamma^0 \Gamma_b \theta
\end{aligned} \tag{7.49}$$

7.4 Final expression for fermionic terms in Hamiltonian

We are now ready to write the final expressions for the fermionic terms in the Hamiltonian. The pp-wave Hamiltonian $\mathcal{H}_{2,F}$ is given by (5.30), as computed in Section 5. Combining (7.11) and (7.47) we see that the cubic piece $\mathcal{H}_{3,BF}$ is given by

$$\begin{aligned}
\mathcal{H}_{3,BF} = & \frac{i}{2} \sum_{i=1}^8 (C_{+i} p_i + \tilde{C}_{+i} X'^i) - \frac{ic}{4} (B_{+56} - B_{+78}) u_4 - \frac{ic}{4} B_{+-4} u_i \\
& - \frac{i}{4} \sum_{i=5}^8 s_i (B_{+4i} p_i + \tilde{B}_{+4i} X'^i) - \frac{i}{8} \sum_{i,j=5}^8 \epsilon_{ij} (B_{+-i} p_j + \tilde{B}_{+-i} X'^j)
\end{aligned} \tag{7.50}$$

Combining (7.12) and (7.48) we see that the two-boson-two-fermion piece $\mathcal{H}_{4,BF}$ is given by

$$\begin{aligned}
\mathcal{H}_{4,BF} = & \frac{i}{c^2} \sum_{i=1}^8 \left(p_i^2 + (X'^i)^2 \right) \left[\tilde{A}_{+,\sigma} + \frac{c}{4} (B_{+56} + B_{+78} - C_{++} + C_{+-}) \right] - i \tilde{A}_{+,\sigma} \left[\sum_{i=1}^3 u_i^2 - u_4^2 \right] \\
& + \frac{2i}{c^2} \sum_{i=1}^8 p_i X'^i \left[A_{+,\sigma} + \frac{c}{4} (\tilde{B}_{+56} + \tilde{B}_{+78}) + \frac{c}{4} \tilde{C}_{+-} \right] + \frac{ic}{2} \sum_{i=1}^3 u_i^2 C_{++} - \frac{ic}{4} \sum_{i=1}^4 u_i^2 (B_{+56} + B_{+78}) \\
& + \frac{i}{2} u_4 \sum_{i=5}^8 s_i \left[C_{+i} p_i - \tilde{C}_{+i} X'^i \right] - \frac{i}{c} \sum_{i,j=1}^8 \left[C_{ij} (X'^i X'^j - p_i p_j) + 2 \tilde{C}_{ij} X'^i p_j \right] - i \sum_{i,j=1}^3 u'_i u_j \tilde{B}_{+ij} \\
& - \frac{i}{8} u_4 \sum_{i,j=5}^8 s_i \epsilon_{ij} (3 B_{+-i} p_j + \tilde{B}_{+-i} X'^j) + \frac{i}{4} (B_{+56} p_{x_1} y_1 + \tilde{B}_{+56} x'_1 y_1 + B_{+78} p_{x_2} y_2 + \tilde{B}_{+78} x'_2 y_2) \\
& - \frac{i}{2} \sum_{i=1}^4 \sum_{j=1}^8 u_i \left[B_{-ij} p_j - \tilde{B}_{-ij} X'^j \right] + \frac{i}{2c} \sum_{i=1}^8 \sum_{j=5}^8 s_j \left[(p_i p_j - X'^i X'^j) B_{4ij} + (p_i X'^j - X'^i p_j) \tilde{B}_{4ij} \right] \\
& - \frac{i}{2} \sum_{i=1}^3 \sum_{j=4}^8 u_i \left[B_{+ij} p_j - \tilde{B}_{+ij} X'^j \right] - \frac{i}{4} u_4 \sum_{i=5}^8 (B_{+4i} p_i + 3 \tilde{B}_{+4i} X'^i) + \frac{i}{2} u_4 \sum_{i=1}^3 (B_{+4i} p_i - \tilde{B}_{+4i} u'_i) \\
& - \frac{i}{4c} \sum_{i=1}^8 \sum_{j,k=5}^8 \epsilon_{jk} \left[(B_{+ij} - B_{-ij}) (p_i p_k - X'^i X'^k) + (\tilde{B}_{+ij} - \tilde{B}_{-ij}) (p_i X'^k - X'^i p_k) \right] \\
& + \frac{i}{2c^2} \sum_{i,j=1}^8 (p_i p'_j + X'^i X''^j) \tilde{E}_{ij} - \frac{i}{2c^2} \sum_{i,j=1}^8 (X'^i p'_j + p_i X''^j) E_{ij} - \frac{3i}{4c} \sum_{i,j=1}^8 (p_i p_j - X'^i X'^j) \mathcal{C}_{i+;j} \\
& + \frac{3i}{4c} \sum_{i,j=1}^8 (X'^i p_j - p_i X'^j) \tilde{\mathcal{C}}_{i+;j} - \frac{i}{4c} \sum_{i,j=1}^8 (p_i p_j + X'^i X'^j) \mathcal{C}_{+i;j} - \frac{i}{4c} \sum_{i,j=1}^8 (X'^i p_j + p_i X'^j) \tilde{\mathcal{C}}_{+i;j} \\
& + \frac{i u_4}{2} \sum_{i=1}^8 (p_j \mathcal{B}_{+-4;i} - X'^j \tilde{\mathcal{B}}_{+-4;i}) + \frac{i}{2c} \sum_{i=5}^8 \sum_{j=1}^8 s_i \left[(p_i p_j - X'^i X'^j) \mathcal{B}_{+4i;j} + (X'^i p_j - p_i X'^j) \tilde{\mathcal{B}}_{+4i;j} \right] \\
& + \frac{i}{4c} \sum_{i,j=5}^8 \sum_{k=1}^8 \epsilon_{ij} \left[(p_i p_k + X'^i X'^k) (\mathcal{B}_{+-i;k} + E_{jk}) + (X'^i p_k + p_i X'^k) (\tilde{\mathcal{B}}_{+-i;k} - \tilde{E}_{jk}) \right]
\end{aligned} \tag{7.51}$$

Here we still use the definitions (4.10) and (7.49) as the definitions of the various two-fermion objects. However, we take θ in these definitions to be given in terms of ψ and ρ as

$$\theta(\psi, \rho) = \frac{1}{2} (\psi + E^{-1} \rho) + \frac{\Gamma_{049}}{2i} (\psi - E^{-1} \rho) \tag{7.52}$$

In this sense we have specified how $\mathcal{H}_{3,BF}$ and $\mathcal{H}_{4,BF}$ in (7.50)-(7.51) depends on the fermionic phase space variables ψ and ρ . Finally, the four-fermion Hamiltonian $\mathcal{H}_{4,F}$ is given by (7.13) since the field redefinition (7.46) does not induce any additional four-fermion terms. Again, the two-fermion objects in (7.13) are defined by (4.10) with $\theta(\psi, \rho)$ given by (7.52).

Thus, we have now specified all the quadratic, cubic and quartic terms in the Hamiltonian in terms of the bosonic and fermionic phase space variables. And, furthermore, by implementing the Dirac procedure as part of a field redefinition we have made sure that the quantized Hamiltonian has the canonical commutation relations for the fermions, *i.e.* that the anti-commutation relation (5.29) is not corrected by $1/R$ or $1/R^2$ corrections.

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A $\text{AdS}_4 \times \mathbb{CP}^3$ background

The $\text{AdS}_4 \times \mathbb{CP}^3$ background has the metric

$$ds^2 = \frac{R^2}{4} \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\hat{\Omega}_2^2 \right) + R^2 ds_{\mathbb{CP}^3}^2 \quad (\text{A.1})$$

with

$$ds_{\mathbb{CP}^3}^2 = \frac{1}{4} d\psi^2 + \frac{1 - \sin \psi}{8} d\Omega_2^2 + \frac{1 + \sin \psi}{8} d\Omega_2'^2 + \cos^2 \psi (d\delta + \omega)^2 \quad (\text{A.2})$$

where the one-form ω and the metrics for the two two-spheres are given by

$$\omega = \frac{1}{4} \sin \theta_1 d\varphi_1 + \frac{1}{4} \sin \theta_2 d\varphi_2 \quad (\text{A.3})$$

$$d\Omega_2^2 = d\theta_1^2 + \cos^2 \theta_1 d\varphi_1^2, \quad d\Omega_2'^2 = d\theta_2^2 + \cos^2 \theta_2 d\varphi_2^2 \quad (\text{A.4})$$

The two-form and four-form field strengths are

$$\frac{1}{R} F_{(2)} = -\cos \psi d\psi \wedge (d\delta + \omega) + \frac{1 - \sin \psi}{4} \cos \theta_1 d\theta_1 \wedge d\varphi_1 - \frac{1 + \sin \psi}{4} \cos \theta_2 d\theta_2 \wedge d\varphi_2 \quad (\text{A.5})$$

$$\frac{1}{R^3} F_{(4)} = \frac{3}{8} \epsilon_{\text{AdS}_4} = \frac{3}{8} \cosh \rho \sinh^2 \rho dt \wedge d\rho \wedge d\hat{\Omega}_2 \quad (\text{A.6})$$

The curvature radius R is given by

$$R^4 = 32\pi^2 \lambda l_s^4 \quad (\text{A.7})$$

Defining the coordinates u_1, u_2 and u_3 by

$$\frac{R}{2} \sinh \rho = \frac{u}{1 - \frac{u^2}{R^2}}, \quad \frac{R^2}{4} (d\rho^2 + \sinh^2 \rho d\hat{\Omega}_2^2) = \frac{\sum_{i=1}^3 du_i^2}{(1 - \frac{u^2}{R^2})^2}, \quad u^2 = \sum_{i=1}^3 u_i^2 \quad (\text{A.8})$$

and the coordinates x_i, y_i , $i = 1, 2$, and u_4 by

$$x_1 = R\varphi_1, \quad y_1 = R\theta_1, \quad x_2 = R\varphi_2, \quad y_2 = R\theta_2, \quad u_4 = \frac{R}{2} \psi \quad (\text{A.9})$$

we can write the metric as

$$ds^2 = -\frac{R^2}{4} \frac{(1 + \frac{u^2}{R^2})^2}{(1 - \frac{u^2}{R^2})^2} dt^2 + \frac{\sum_{i=1}^3 du_i^2}{(1 - \frac{u^2}{R^2})^2} + du_4^2 + R^2 \cos^2 \frac{2u_4}{R} \left[d\delta^2 + 2d\delta \left(\sin \frac{y_1}{R} \frac{dx_1}{4R} + \sin \frac{y_2}{R} \frac{dx_2}{4R} \right) \right] \\ + \frac{1}{8} \left(\cos \frac{u_4}{R} - \sin \frac{u_4}{R} \right)^2 \left(dy_1^2 + \cos^2 \frac{y_1}{R} dx_1^2 \right) + \frac{1}{8} \left(\cos \frac{u_4}{R} + \sin \frac{u_4}{R} \right)^2 \left(dy_2^2 + \cos^2 \frac{y_2}{R} dx_2^2 \right) \quad (\text{A.10})$$

This corresponds to the zehnbeins

$$e^0 = \frac{R}{2} \frac{1 + \frac{u^2}{R^2}}{1 - \frac{u^2}{R^2}} dt, \quad e^i = \frac{du_i}{1 - \frac{u^2}{R^2}}, \quad i = 1, 2, 3 \quad (\text{A.11})$$

$$e^5 = \frac{1}{2\sqrt{2}} \left(\cos \frac{u_4}{R} - \sin \frac{u_4}{R} \right) \cos \frac{y_1}{R} dx_1, \quad e^6 = \frac{1}{2\sqrt{2}} \left(\cos \frac{u_4}{R} - \sin \frac{u_4}{R} \right) dy_1 \quad (\text{A.12})$$

$$e^7 = \frac{1}{2\sqrt{2}} \left(\cos \frac{u_4}{R} + \sin \frac{u_4}{R} \right) \cos \frac{y_2}{R} dx_2, \quad e^8 = \frac{1}{2\sqrt{2}} \left(\cos \frac{u_4}{R} + \sin \frac{u_4}{R} \right) dy_2 \quad (\text{A.13})$$

$$e^4 = du_4, \quad e^9 = \frac{R}{2} \cos \frac{2u_4}{R} \left[2d\delta + \frac{1}{2R} \left(\sin \frac{y_1}{R} dx_1 + \sin \frac{y_2}{R} dx_2 \right) \right] \quad (\text{A.14})$$

Using these the two-form and four-form field strengths takes the form

$$F_{(2)} = \frac{2}{R} (-e^4 \wedge e^9 - e^5 \wedge e^6 + e^7 \wedge e^8) \quad (\text{A.15})$$

$$F_{(4)} = \frac{6}{R} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \quad (\text{A.16})$$

We make the coordinate transformation

$$\delta = \frac{1}{2}t + \frac{v}{R^2} \quad (\text{A.17})$$

Written explicitly, the metric in these coordinates becomes

$$ds^2 = -dt^2 \left(\frac{R^2}{4} \sin^2 \frac{2u_4}{R} + \frac{u^2}{(1 - \frac{u^2}{R^2})^2} \right) + \frac{\sum_{i=1}^3 du_i^2}{(1 - \frac{u^2}{R^2})^2} + du_4^2 \\ + \frac{1}{8} \left(\cos \frac{u_4}{R} - \sin \frac{u_4}{R} \right)^2 \left(dy_1^2 + \cos^2 \frac{y_1}{R} dx_1^2 \right) + \frac{1}{8} \left(\cos \frac{u_4}{R} + \sin \frac{u_4}{R} \right)^2 \left(dy_2^2 + \cos^2 \frac{y_2}{R} dx_2^2 \right) \\ + R^2 \cos^2 \frac{2u_4}{R} \left[dt + \frac{dv}{R^2} + \sin \frac{y_1}{R} \frac{dx_1}{4R} + \sin \frac{y_2}{R} \frac{dx_2}{4R} \right] \left[\frac{dv}{R^2} + \sin \frac{y_1}{R} \frac{dx_1}{4R} + \sin \frac{y_2}{R} \frac{dx_2}{4R} \right] \quad (\text{A.18})$$

Define

$$e^+ = \frac{1}{2R} (e^0 + e^9), \quad e^- = \frac{1}{2} (e^0 - e^9) \quad (\text{A.19})$$

B Gamma-matrix conventions

Define the real 8×8 matrices $\gamma_1, \dots, \gamma_8$ as in [19]. They obey

$$\begin{aligned} \gamma_i \gamma_j^T + \gamma_j \gamma_i^T &= \gamma_i^T \gamma_j + \gamma_j^T \gamma_i = 2\delta_{ij} I_8, \quad i, j = 1, \dots, 8 \\ \gamma_1 \gamma_2^T \gamma_3 \gamma_4^T \gamma_5 \gamma_6^T \gamma_7 \gamma_8^T &= I_8, \quad \gamma_1^T \gamma_2^T \gamma_3^T \gamma_4^T \gamma_5^T \gamma_6^T \gamma_7^T \gamma_8^T = -I_8 \end{aligned} \quad (\text{B.1})$$

where I_n is the $n \times n$ identity matrix. Define the 16×16 matrices $\hat{\gamma}_1, \dots, \hat{\gamma}_9$ by

$$\hat{\gamma}_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i^T & 0 \end{pmatrix}, \quad i = 1, \dots, 8, \quad \hat{\gamma}_9 = \begin{pmatrix} I_8 & 0 \\ 0 & -I_8 \end{pmatrix} \quad (\text{B.2})$$

The matrices $\hat{\gamma}_1, \dots, \hat{\gamma}_9$ are symmetric and real and they obey

$$\{\hat{\gamma}_i, \hat{\gamma}_j\} = 2\delta_{ij} I_{16}, \quad i, j = 1, \dots, 9, \quad \hat{\gamma}_9 = \hat{\gamma}_1 \hat{\gamma}_2 \cdots \hat{\gamma}_8 \quad (\text{B.3})$$

Define the 32×32 matrices

$$\Gamma_0 = \begin{pmatrix} 0 & -I_{16} \\ I_{16} & 0 \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} 0 & \hat{\gamma}_i \\ \hat{\gamma}_i & 0 \end{pmatrix}, \quad i = 1, \dots, 9, \quad \Gamma_{11} = \begin{pmatrix} I_{16} & 0 \\ 0 & -I_{16} \end{pmatrix} \quad (\text{B.4})$$

These matrices are real and obey

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} I_{32}, \quad i, j = 0, 1, \dots, 9, 11, \quad \Gamma_{11} = \Gamma^0 \Gamma^1 \cdots \Gamma^9 \quad (\text{B.5})$$

We define

$$\gamma_{i_1 \cdots i_{2k}} = \gamma_{[i_1} \gamma_{i_2}^T \cdots \gamma_{i_{2k}}^T], \quad \gamma_{i_1 i_2 \cdots i_{2k+1}} = \gamma_{[i_1}^T \gamma_{i_2} \cdots \gamma_{i_{2k+1}}^T], \quad i_l = 1, \dots, 8 \quad (\text{B.6})$$

$$\hat{\gamma}_{i_1 \cdots i_n} = \hat{\gamma}_{[i_1} \hat{\gamma}_{i_2} \cdots \hat{\gamma}_{i_n]}, \quad i_l = 1, \dots, 9 \quad (\text{B.7})$$

$$\Gamma_{i_1 i_2 \cdots i_n} = \Gamma_{[i_1} \Gamma_{i_2} \cdots \Gamma_{i_n]}, \quad i_l = 0, 1, \dots, 9, 11 \quad (\text{B.8})$$

C Structure constants and \mathcal{M}^2

We can write the $OSp(6|2, 2)$ algebra schematically on the form

$$[B_i, B_j] = f_{ij}^k B_k, \quad [F_\alpha, B_i] = \tilde{f}_{\alpha i}^\beta F_\beta, \quad \{F_\alpha, F_\beta\} = \hat{f}_{\alpha\beta}^i B_i \quad (\text{C.1})$$

where B_i are the bosonic generators which generate an $SO(2, 3) \times SU(4)$ algebra (with $25 = 10 + 15$ generators) and F_α corresponds to the 24 fermionic generators. We take α to run over all 32 fermionic directions. The 24 fermionic generators are then defined by $P_{\alpha\beta} F_\beta = F_\alpha$.

The structure constants

The structure constants $\tilde{f}_{\alpha i}^\beta$ can be read off from the covariant derivative (2.5)⁵

$$\tilde{f}_{\beta a}^\alpha = \frac{1}{R}(\Gamma_{0123}P\Gamma_a P)^\alpha_\beta, \quad \tilde{f}_{\beta \hat{a}\hat{b}}^\alpha = -\frac{1}{4}(P\Gamma_{\hat{a}\hat{b}}P)^\alpha_\beta, \quad \tilde{f}_{\beta a'b'}^\alpha = -\frac{1}{4}(P\Gamma_{a'b'}P)^\alpha_\beta \quad (\text{C.2})$$

where $a = 0, 1, \dots, 9$, $\hat{a}, \hat{b} = 0, 1, 2, 3$ and $a', b' = 4, 5, \dots, 9$. The structure constants $\hat{f}_{\alpha\beta}^i$ are instead

$$\begin{aligned} \hat{f}_{\alpha\beta}^a &= 2i(P\Gamma^0\Gamma^a P)_{\alpha\beta} \\ \hat{f}_{\alpha\beta}^{\hat{a}\hat{b}} &= -\frac{4i}{R}(P\Gamma^0\Gamma_{0123}\Gamma^{\hat{a}\hat{b}}P)_{\alpha\beta} \\ \hat{f}_{\alpha\beta}^{a'b'} &= \frac{2i}{R}\left(P\Gamma^0\left(\Gamma_{0123}\Gamma^{a'b'} - J^{a'b'}\Gamma_{11}\right)P\right)_{\alpha\beta} \end{aligned} \quad (\text{C.3})$$

where $a = 0, 1, \dots, 9$, $\hat{a}, \hat{b} = 0, 1, 2, 3$ and $a', b' = 4, 5, \dots, 9$, and where we introduced the Kaehler form

$$J^{a'b'} = \delta^{a'7}\delta^{b'8} - \delta^{a'8}\delta^{b'7} - \delta^{a'4}\delta^{b'9} + \delta^{a'9}\delta^{b'4} - \delta^{a'5}\delta^{b'6} + \delta^{a'6}\delta^{b'5} \quad (\text{C.4})$$

For $a' = 4, \dots, 9$ the structure constants $\hat{f}_{\alpha\beta}^{a'}$ can also be written as

$$\hat{f}_{\alpha\beta}^{a'} = i(P\Gamma^0\Gamma^{a'}P)_{\alpha\beta} + i(P\Gamma^0\Gamma_{0123}\Gamma_{11}J^{a'b'}\Gamma_{b'}P)_{\alpha\beta} \quad (\text{C.5})$$

using here the relation

$$(P\Gamma^{a'}P)_{\alpha\beta} = (P\Gamma_{0123}\Gamma_{11}J^{a'b'}\Gamma_{b'}P)_{\alpha\beta} \quad (\text{C.6})$$

The fermionic matrix \mathcal{M}^2

We now determine the fermionic matrix \mathcal{M}^2 needed for writing the four-fermion terms in the Lagrangian of Section 2. It can generally be written in terms of structure constants of the $OSp(6|2, 2)$ algebra (C.1) as

$$(\mathcal{M}^2)_\beta^\alpha = -\theta^\gamma \tilde{f}_{\gamma i}^\alpha \theta^\delta \hat{f}_{\delta\beta}^i = -\theta^\gamma \tilde{f}_{\gamma a}^\alpha \theta^\delta \hat{f}_{\delta\beta}^a - \theta^\gamma \tilde{f}_{\gamma ab}^\alpha \theta^\delta \hat{f}_{\delta\beta}^{ab} \quad (\text{C.7})$$

Using the structure constants written above we compute

$$\begin{aligned} (\mathcal{M}^2)_\beta^\alpha &= -\frac{2i}{R}(P\Gamma_{0123}\Gamma_a P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma^a P)_{\delta\beta} \\ &\quad - \frac{i}{R}(P\Gamma_{\hat{a}\hat{b}}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma^{\hat{a}\hat{b}}P)_{\delta\beta} \\ &\quad + \frac{i}{2R}(P\Gamma_{a'b'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0(\Gamma_{0123}\Gamma^{a'b'} - J^{a'b'}\Gamma_{11})P)_{\delta\beta} \end{aligned} \quad (\text{C.8})$$

where $\hat{a}, \hat{b} = 0, 1, 2, 3$ and $a', b' = 4, 5, \dots, 9$. This can also be written in the form

$$(\mathcal{M}^2)_\beta^\alpha = -\frac{2i}{R}(P\Gamma_{0123}\Gamma_{\hat{a}}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma^{\hat{a}}P)_{\delta\beta} - \frac{i}{R}(P\Gamma_{0123}\Gamma_{a'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma^{a'}P)_{\delta\beta}$$

⁵Note that $\tilde{f}_{\beta a}^\alpha$ for $a = 0, 1, 2, 3$ corresponds to four of the $SO(2, 3)$ generators, the other six corresponding to $\tilde{f}_{\beta \hat{a}\hat{b}}^\alpha$ since this is antisymmetric in \hat{a}, \hat{b} . Similarly, the $\tilde{f}_{\beta a}^\alpha$ for $a = 4, 5, \dots, 9$ corresponds to six of the $SU(4)$ generators. This leaves the last 9 for $\tilde{f}_{\beta a'b'}^\alpha$, despite the fact that antisymmetry of a', b' seemingly gives 15. However, the projector P in $\tilde{f}_{\beta a'b'}^\alpha$ gives relations between the matrices therefore only 9 of them are independent. Thus, we get the 15 generators of $SU(4)$. The same story is true for $\hat{f}_{\alpha\beta}^i$.

$$\begin{aligned}
& + \frac{i}{R} (P\Gamma_{11}\Gamma_{a'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma_{11}\Gamma_{a'}P)_{\delta\beta} - \frac{i}{R} (P\Gamma_{\hat{a}\hat{b}}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma^{\hat{a}\hat{b}}P)_{\delta\beta} \\
& + \frac{i}{2R} (P\Gamma_{a'b'}P)^\alpha_\gamma \theta^\gamma \theta^\delta (P\Gamma^0\Gamma_{0123}\Gamma^{a'b'}P)_{\delta\beta} - \frac{i}{R} (\Gamma_{0123}\Gamma_{11})^\alpha_\gamma \theta^\gamma \theta^\delta (\Gamma^0\Gamma_{11})_{\delta\beta}
\end{aligned} \tag{C.9}$$

where we eliminated $J^{a'b'}$ by taking into account that

$$J = \frac{1}{2} \Gamma_{0123} \Gamma_{11} J^{a'b'} \Gamma_{a'b'} \tag{C.10}$$

along with the relation (C.6) and that J on supersymmetric fermions gives $J\theta = -\theta$.

Equivalence with \mathcal{M}^2 in alternative representation

We now show that the formula (C.9) is equivalent to the one written in [22, 23]. Thus, we shall use the following alternative representation of the Gamma matrices [22, 23]

$$\begin{aligned}
\Gamma^{\hat{a}} &= \gamma^{\hat{a}} \otimes \mathbf{1}, & \Gamma^{a'} &= \gamma^5 \otimes \gamma^{a'}, & \Gamma^{11} &= \gamma^5 \otimes \gamma^7, \\
\hat{a} &= 0, 1, 2, 3; & a' &= 4, \dots, 9.
\end{aligned}$$

Here $(\gamma^{\hat{a}})_{\hat{\alpha}\hat{\beta}}$ are 4-dimensional matrices, corresponding to the AdS_4 part, $\hat{\alpha}, \hat{\beta} = 1, \dots, 4$ and $(\gamma^{a'})_{\alpha'\beta'}$ are 8-dimensional matrices, $\alpha', \beta' = 1, \dots, 8$, corresponding to the 6-dimensional space \mathbb{CP}^3 . Eq.(C.6) becomes

$$P_6 \gamma^{a'} P_6 = i P_6 J^{a'b'} \gamma_{b'} \gamma^7 P_6 \tag{C.11}$$

this was derived in [22]. Here P_6 is the reduction of P to \mathbb{CP}^3

$$P_6 = \frac{3-J}{4}, \quad 2J = -i J^{a'b'} \gamma_{a'b'} \gamma^7. \tag{C.12}$$

This projector when acting on an 8-dimensional spinor annihilates 2 and leaves 6 of its components. Thus the spinor

$$\theta^{\hat{\alpha}\alpha'} = (P_6 \theta)^{\hat{\alpha}\alpha'} \iff \theta^{\hat{\alpha}a'} \quad a' = 1, \dots, 6 \tag{C.13}$$

has 24 non-zero components. In terms of the dimensionally reduced γ -matrices \mathcal{M}^2 reads

$$\begin{aligned}
(\mathcal{M}^2)^\alpha_\beta &= -\frac{2}{R} (\gamma^5 \gamma_{\hat{a}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}b'}_{\gamma^0 \gamma^{\hat{a}}} (\gamma^0 \gamma^{\hat{a}})_{\hat{\delta}\hat{\beta}} - \frac{1}{R} (\gamma_{c'})^{a'}_{d'} \theta^{\hat{\alpha}d'} \theta^{\hat{\delta}}_{f'} (\gamma^0 \gamma^5)_{\hat{\delta}\hat{\beta}} (\gamma^{c'})^{f'}_{b'} \\
&- \frac{i}{R} (\gamma_{c'})^{a'}_{d'} \theta^{\hat{\alpha}d'} \theta^{\hat{\delta}}_{f'} (\gamma^0 \gamma^5)_{\hat{\delta}\hat{\beta}} (J^{c'g'} \gamma_{g'} \gamma^7)^{f'}_{b'} - \frac{1}{R} (\gamma_{\hat{a}\hat{b}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}b'}_{\gamma^0 \gamma^5 \gamma^{\hat{a}\hat{b}}} (\gamma^0 \gamma^5 \gamma^{\hat{a}\hat{b}})_{\hat{\delta}\hat{\beta}} \\
&+ \frac{1}{2R} (\gamma_{c'g'})^{a'}_{d'} \theta^{\hat{\alpha}d'} \theta^{\hat{\delta}}_{f'} (\gamma^0 \gamma^5)_{\hat{\delta}\hat{\beta}} (\gamma^{c'g'})^{f'}_{b'} - \frac{i}{2R} (\gamma_{c'g'})^{a'}_{d'} \theta^{\hat{\alpha}d'} \theta^{\hat{\delta}}_{f'} (\gamma^0 \gamma^5)_{\hat{\delta}\hat{\beta}} (J^{c'g'} \gamma^7)^{f'}_{b'}
\end{aligned} \tag{C.14}$$

Using the relations

$$\gamma_{c'} J^{c'b'} = i \gamma^{g'} \gamma^7, \quad J^{c'g'} \gamma_{c'g'} \theta = -2i \gamma^7 \theta \tag{C.15}$$

we find

$$(\mathcal{M}^2)^\alpha_\beta = -\frac{2}{R} (\gamma^5 \gamma_{\hat{a}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}b'}_{\gamma^0 \gamma^{\hat{a}}} (\gamma^0 \gamma^{\hat{a}})_{\hat{\delta}\hat{\beta}} - \frac{1}{R} (\gamma_{\hat{a}\hat{b}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}b'}_{\gamma^0 \gamma^5 \gamma^{\hat{a}\hat{b}}} (\gamma^0 \gamma^5 \gamma^{\hat{a}\hat{b}})_{\hat{\delta}\hat{\beta}}$$

$$\begin{aligned}
& -\frac{1}{R} \left[(\gamma^{c'})^{a'}_{d'} (\gamma^{c'})^{f'}_{b'} - (\gamma^7 \gamma^{g'})^{a'}_{d'} (\gamma^7 \gamma^{g'})^{f'}_{b'} - \frac{1}{2} (\gamma^{c'g'})^{a'}_{d'} (\gamma^{c'g'})^{f'}_{b'} \right. \\
& \left. + (\gamma^7)^{a'}_{d'} (\gamma^7)^{f'}_{b'} \right] \theta^{\hat{\alpha}d'} \theta^{\hat{\delta}}_{f'} (\gamma^0 \gamma^5)_{\hat{\delta}\hat{\beta}}
\end{aligned} \tag{C.16}$$

We can now use the Fierz identity for the 8 dimensional gamma matrices $\gamma^{a'}$ in 6 dimensions

$$\begin{aligned}
(\gamma_{a'})_{\alpha\beta} (\gamma^{a'})_{\gamma\delta} &= 4(\delta_{\alpha\delta} \delta_{\beta\gamma} - \delta_{\alpha\gamma} \delta_{\beta\delta}) + \frac{1}{2} (\gamma_{a'b'})_{\alpha\beta} (\gamma^{a'b'})_{\gamma\delta} \\
&+ (\gamma^7 \gamma_{a'})_{\alpha\beta} (\gamma^7 \gamma^{a'})_{\gamma\delta} - (\gamma^7)_{\alpha\beta} (\gamma^7)_{\gamma\delta}
\end{aligned} \tag{C.17}$$

giving

$$\begin{aligned}
(\mathcal{M}^2)_{\beta}^{\alpha} &= -\frac{2}{R} (\gamma^5 \gamma_{\hat{a}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}}_{b'} (\gamma^0 \gamma^{\hat{a}})_{\hat{\delta}\hat{\beta}} - \frac{1}{R} (\gamma_{\hat{a}\hat{b}})^{\hat{\alpha}}_{\hat{\gamma}} \theta^{\hat{\gamma}a'} \theta^{\hat{\delta}}_{b'} (\gamma^0 \gamma^5 \gamma^{\hat{a}\hat{b}})_{\hat{\delta}\hat{\beta}} \\
&+ \frac{4}{R} \theta^{\hat{a}}_{b'} (\theta^{a'} \gamma^5)_{\hat{\beta}} - \frac{4}{R} \delta^{a'}_{b'} \theta^{\hat{a}c'} (\theta \gamma^5)_{\hat{\beta}c'}
\end{aligned} \tag{C.18}$$

which is the \mathcal{M}^2 found in Ref.[22, 23].

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